MATH 1271: Calculus I

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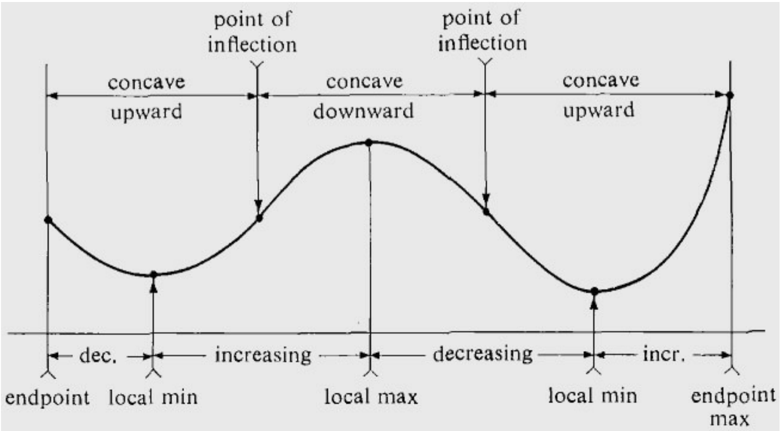
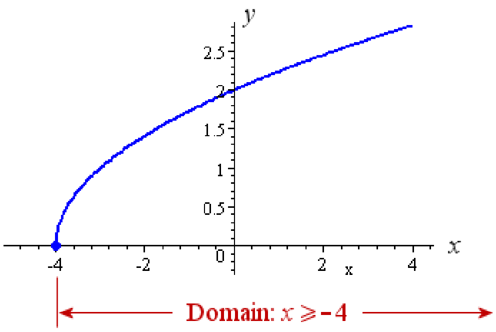
4.5 - Summary of Curve Sketching

Review



Given a function $f(x)$, how might one go about trying to sketch it?
That's what we explore below.

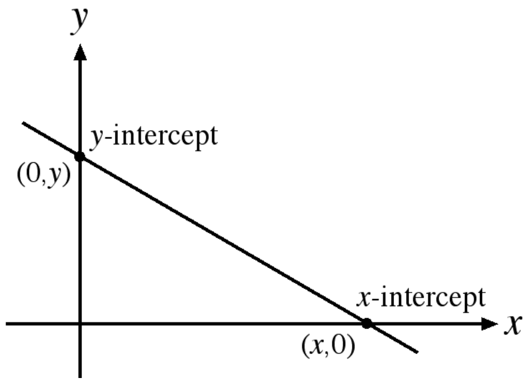
A. **Domain**: It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.



B. **Intercepts**:

i) The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis.

ii) To find the x-intercepts, we set $y = 0$ and solve for x.
(You could omit this step if the equation is difficult to solve.)

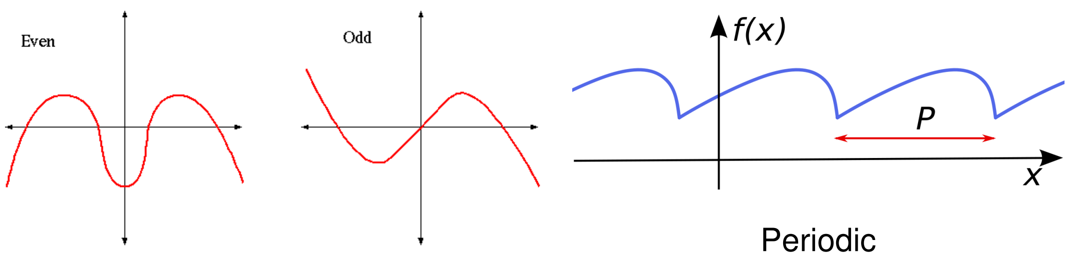


C. **Symmetry**:

i) If $f(-x) = f(x)$ for all x in D, then f is an **even function** and the curve is symmetric across the y-axis.

ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric across the origin.

iii) If $f(x+P) = f(x)$ for all x in D, where P is a positive constant, then f is called a **periodic function** and the smallest such number P is called its **period**.

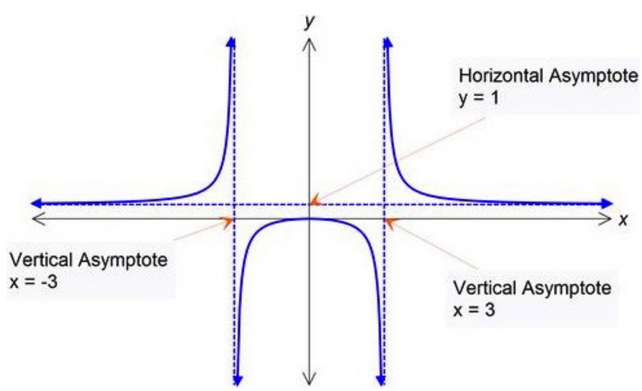


Periodic

D. **Asymptotes**:

i) **Horizontal Asymptotes.** If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve. Also, knowing that $\lim_{x\to\pm\infty} f(x) = \pm\infty$ helps you to graph the curve (even though there is no asymptote).

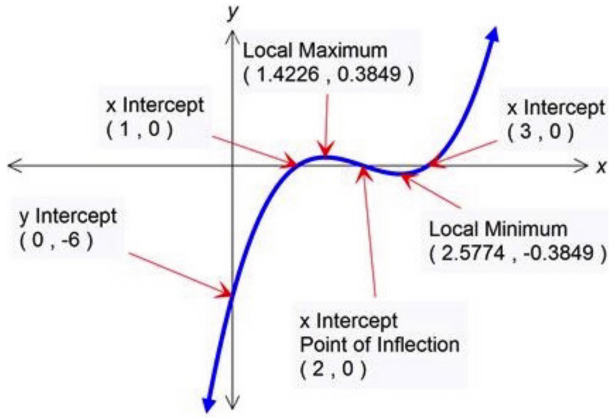
ii) **Vertical Asymptotes.** The line $x = a$ is a vertical asymptote if at least one of the following statements is true: $\lim_{x=a^+} f(x) = \pm\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$.
This is most often discovered by observing that the denominator goes to ∞ when x approaches some number (for example, $\frac{1}{x-a} \to \pm\infty$ when x approaches a).



E/F. **Intervals of Increase or Decrease and Max/Mins**: Find the critical numbers of f. Find the intervals on which $f'(x)$ is positive and the intervals on which it is negative. This will also locate the maximum and minimum values.

G. **Concavity and Points of Inflection**: Compute $f''(x)$ and use the concavity test to locate and draw the points of inflection.

H. **Sketch of the Curve**: Use the information above to draw the graph.
Draw the asymptotes as dashed lines.
Plot the intercepts, maximum and minimum points, and inflection points.
Be careful when drawing the curve to generate the inflections
at the appropriate points, and respect the asymptotes.



Problem 14. Use the guidelines of this section to sketch $f = \frac{x^2}{x^2+9}$.

A. **Domain**

$= \mathbb{R}$.

B. $\frac{x^2}{x^2+9}$ **Find the y-intercept and x-intercept**:

y-intercept: $f(0) = 0$, so $(0,0)$; x-intercept: $f(x) = 0$ when $x = 0$, so $(0,0)$.

C. $\frac{x^2}{x^2+9}$ **Symmetric**?

$f(-x) = f(x)$, so f is even and symmetric around the y-axis.

D. $\frac{x^2}{x^2+9}$ **HA; VA**?

$$\lim_{x\to\pm\infty}\left[\frac{x^2}{x^2+9}\right] = \lim_{x\to\pm\infty}\left[\frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}}\right] = \frac{1}{1+\lim\limits_{x\to\pm\infty}\frac{9}{x^2}} = 1, \text{ so } y = 1 \text{ is a HA.}$$

No VA (since $x^2 + 9$ can't be equal to zero)

E. $\frac{x^2}{x^2+9}$ **Increasing, decreasing intervals.**

$$f'(x) = \frac{(x^2+9)(2x)-x^2(2x)}{(x^2+9)^2} = \frac{18x}{(x^2+9)^2}$$

$f'(x) > 0$ when $x > 0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

F. $\frac{x^2}{x^2+9}$ **Local minimum/maximum.**

Local minimum value $f(0) = 0$; no local maximum.

G. $\frac{x^2}{x^2+9}$ **CU, CD, inflection points?**

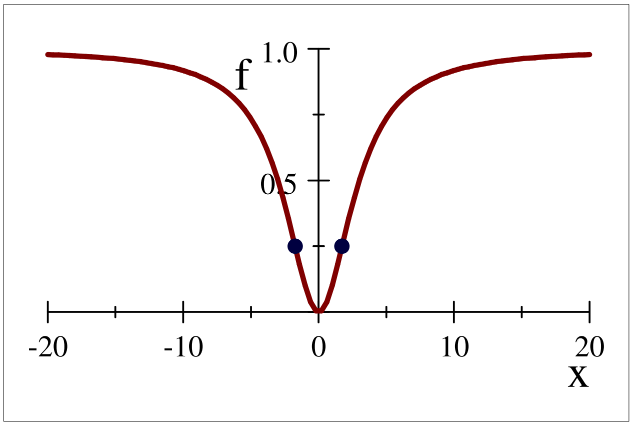
$$f''(x) = \frac{(x^2+9)^2(18)-18x\cdot 2(x^2+9)\cdot 2x}{\left[(x^2+9)^2\right]^2} = \frac{18(x^2+9)[(x^2+9)-4x^2]}{(x^2+9)^4}$$

$$= \frac{18(9-3x^2)}{(x^2+9)^3} = \frac{-54\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{(x^2+9)^3} > 0, \text{ where ??}$$

Observe that the denominator is always positive, and -54 is negative, so if the fraction is going to be greater than zero, we must have $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right) < 0$. Also, observe that interesting things happen when $x = \pm\sqrt{3}$, namely that $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right) = 0$. So, testing points less than $-\sqrt{3}$, greater than $\sqrt{3}$, and in between these values ($x = 0$ is a convenient point to check), we discover that $f''(x) > 0$ only when $-\sqrt{3} < x < \sqrt{3}$.

So f is CU on $\left(-\sqrt{3},\sqrt{3}\right)$ and CD on $\left(-\infty,-\sqrt{3}\right)$ and $\left(\sqrt{3},\infty\right)$.

There are two inflection points: $\left(\pm\sqrt{3}, f\left(\pm\sqrt{3}\right)\right) = \left(\pm\sqrt{3}, \frac{\left(\pm\sqrt{3}\right)^2}{\left(\pm\sqrt{3}\right)^2+9}\right) \approx (\pm 1.73, \frac{1}{4})$.



Problem 32. Use the guidelines of this section to sketch $f = \sqrt[3]{x^3+1}$

A. **Domain**

$= \mathbb{R}$.

B. $\sqrt[3]{x^3+1}$ **y-intercept**? **x-intercept**?

y-intercept: $f(0) = 1,$ so $(0, 1)$;

x-intercept: $f(x) = 0$ when $x^3 + 1 = 0,$ or when $x = -1,$ so $(-1, 0)$.

C. $\sqrt[3]{x^3+1}$ **Symmetric**?

$f(-x) = \sqrt[3]{-x^3+1} \neq \sqrt[3]{x^3+1} = f(x),$ so f is not even.

Also, $f(-x) = \sqrt[3]{-x^3+1} \neq -\sqrt[3]{x^3+1} = -f(x),$ so f is not odd.

No symmetry.

D. $\sqrt[3]{x^3+1}$ **HA; VA**?

$\lim_{x \to \pm\infty} \sqrt[3]{x^3+1} = \pm\infty,$ so no HA.

$\sqrt[3]{x^3+1}$ is continuous on $\mathbb{R}$, so no VA.

No asymptotes.

E. $\sqrt[3]{x^3+1}$ **Increasing, decreasing intervals.**

$$f'(x) = \left(\sqrt[3]{x^3+1}\right)' = \left((x^3+1)^{\frac{1}{3}}\right)' = \tfrac{1}{3}(x^3+2)^{-\frac{2}{3}}(3x^2)$$

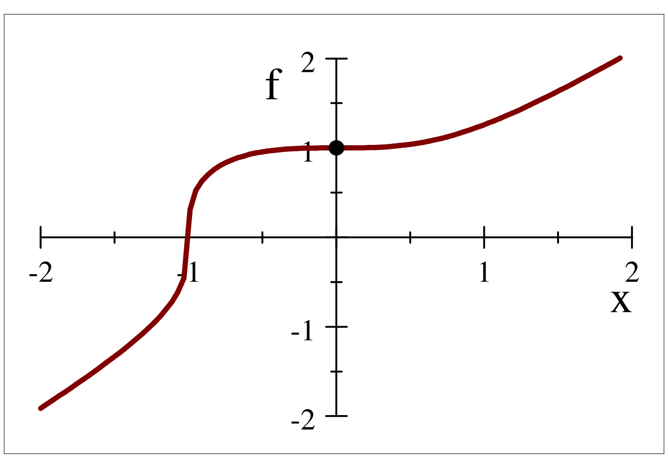
$$= \frac{x^2}{\sqrt[3]{(x^3+1)^2}}.$$

Note that $(x^3+1)^2 \geq 0,$ so the denominator is always nonnegative, and indeed is only zero at $x = -1.$ And the numerator is positive, with the only exception of $x = 0.$ So we only need be worried at the points $\{0,-1\}$. However, note that f' does not change sign at $x = \{0,-1\}$, so f is increasing on $\mathbb{R}$.

F. $\sqrt[3]{x^3+1}$ **Local minimum/maximum.**

$$f' = \frac{x^2}{(x^3+1)^{\frac{2}{3}}}.$$

So we have a critical point $f' = 0$ at $x = 0,$ but we showed above that f' to the left and right of this critical point is positive, so this is not a minimum or maximum, but an inflection point. So, no local extrema.



G. $\sqrt[3]{x^3+1}$ **CU, CD, inflection points?**

$$f''(x) = \left(\frac{x^2}{(x^3+1)^{\frac{2}{3}}}\right)' = \frac{(2x)(x^3+1)^{\frac{2}{3}} - x^2\left[\frac{2}{3}(x^3+1)^{-\frac{1}{3}}(3x^2)\right]}{\left[(x^3+1)^{\frac{2}{3}}\right]^2} = \frac{2x(x^3+1)^{-\frac{1}{3}}[(x^3+1)-x^3]}{(x^3+1)^{\frac{4}{3}}}$$

$$= \frac{2x}{(x^3+1)^{\frac{5}{3}}}.$$

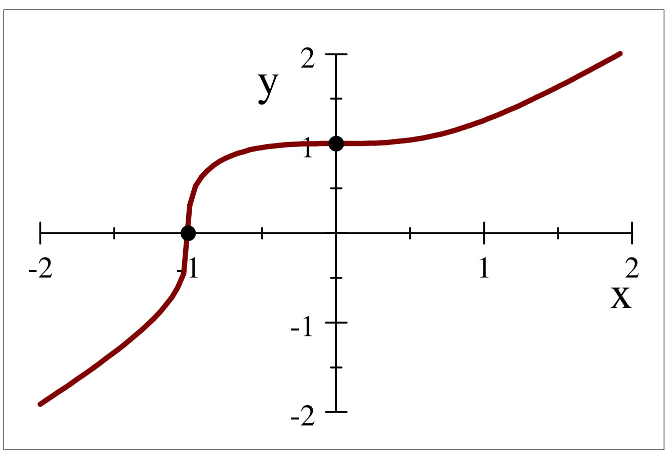
Observe that the numerator changes sign at $x = 0$, and the denominator changes sign at $x = -1$, so it is at these points where the sign of f'' may change.

$f''(x) > 0$ when $x < -1$ or $x > 0$ and

$f''(x) < 0$ when $-1 < x < 0$.

So f is CU on $(-\infty, -1)$, CD on $(-1, 0)$ and CU on $(0, \infty)$.

And we have an inflection point (IP) at $(-1, f(-1)) = (-1, 0)$ and $(0, f(0)) = (0, 1)$.



Problem 52. Use the guidelines of this section to sketch $f = \frac{\ln x}{x^2}$

A. $D = (0, \infty)$

B. **y-intercept**: none; **x-intercept**: $f(x) = 0$ when $\ln x = 0$ or when $x = 1$.

C. **No symmetry**

D. $\lim\limits_{x\to 0^+} f(x) = -\infty$, so $x = 0$ is a VA; $\lim\limits_{x\to\infty} \frac{\ln x}{x^2} \Rightarrow H \Rightarrow \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{2x} = 0$, so $y = 0$ is a HA.

E. $f'(x) = \frac{x^2\left(\frac{1}{x}\right)-(\ln x)(2x)}{(x^2)^2} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$.

$f'(x) > 0 \Leftrightarrow 1 - 2\ln x > 0 \Leftrightarrow \ln x < \frac{1}{2} \Rightarrow 0 < x < e^{\frac{1}{2}}$ and $f'(x) < 0 \Rightarrow x > e^{\frac{1}{2}}$, so f is increasing on $(0, \sqrt{e})$ and decreasing on $(\sqrt{e}, \infty)$.

F. Local maximum value $f\left(e^{\frac{1}{2}}\right) = \frac{\frac{1}{2}}{e} = \frac{1}{2e}$.

G. $f''(x) = \frac{x^3\left(-\frac{2}{x}\right)-(1-2\ln x)(3x^3)}{(x^3)^2} = \frac{x^2[-2-3(1-2\ln x)]}{x^6} = \frac{-5+6\ln x}{x^4}$.

$f''(x) > 0$ when $-5 + 6\ln x > 0 \Leftrightarrow \ln x > \frac{5}{6} \Rightarrow x > e^{\frac{5}{6}}$ (f is CU) and $f''(x) < 0 \Leftrightarrow 0 < x < e^{\frac{5}{6}}$ (f is CD). IP at $\left(e^{\frac{5}{6}}, f\left(e^{\frac{5}{6}}\right)\right) = \left(e^{\frac{5}{6}}, \frac{5}{6e^{\frac{5}{3}}}\right)$.