MATH 1271: Calculus I

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4.4 - L'Hospital's Rule & Intermediate Forms

Review:

Recall how we struggled to solve something like $\lim_{x\to a} \frac{f(x)}{g(x)}$ if the result of substituting $x \to a$ into the fraction resulted in $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$? Now that we know more about derivatives, we have a tool to help us with this problem.

L'Hopital's Rule: We can evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ when we know the following:

- f, g are differentiable.
- On an open interval I that contains a, we have $g'(x) \neq 0$ (except possibly at a).
- And either: $\lim_{x\to a} f(x) = 0$ **and** $\lim_{x\to a} g(x) = 0$,

  or instead that $\lim_{x\to a} f(x) = \pm\infty$ **and** $\lim_{x\to a} g(x) = \pm\infty$.

  (In other words, we have an "**indeterminate form**" of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$).

If these three things are true, then: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

(that is, if the limit on the right side the equation exists (or is ∞ or $-\infty$), then the limit on the left is equal to that same value).

Indeterminate Products: What if $\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) =$ " $0 \cdot \infty$ " ?!?

Transform this into a form where you can use L'Hospital's Rule.

$\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} \frac{f}{\frac{1}{g}} =$ " $\frac{0}{0}$ " or $\lim_{x\to a} \frac{g}{\frac{1}{f}} =$ " $\frac{\infty}{\infty}$ ".

Indeterminate Differences: What if $\lim_{x\to a} [f(x) - g(x)] =$ " $\infty - \infty$ " ?!?

Transform this into a form where you can use L'Hospital's Rule.
You will have to do this using a common denominator (if they are fractions), rationalizing, or factoring out a common factor. The method will be dictated by the specifics of $f(x)$ and $g(x)$ (see examples below).

Indeterminate Powers: What if $\lim_{x\to a} [f(x)]^{g(x)} =$ " 0^0 or ∞^0 or 1^∞ " ?!?

Either take the natural logarithm, or write the function as an exponential.

$\ln y = \ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$ or $[f(x)]^{g(x)} = e^{\ln[f(x)]^{g(x)}} = e^{g(x)\ln[f(x)]}$.

Then try to take your limit to determine the answer: (see examples below).

Problem: **14**. Find $\lim_{x\to 0} \frac{x^2}{1-\cos x}$. Use L'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If L'Hospital's Rule doesn't apply, explain why.

This limit has the form $\frac{0}{0}$.

$\mathbb{H} \Rightarrow \lim_{x \to 0} \frac{(x^2)'}{(1-\cos x)'} = \lim_{x \to 0} \frac{2x}{\sin x}$. This limit again has the form $\frac{0}{0}$.
("$\mathbb{H}$" means I'm applying L'Hospital's rule)

$\mathbb{H} \Rightarrow \lim_{x \to 0} \frac{(2x)'}{(\sin x)'} = \lim_{x \to 0} \frac{2}{\cos x}$,

$= \frac{2}{1} = 2.$ Therefore, $\lim_{x \to 0} \frac{x^2}{1-\cos x} = 2.$

Problem: 38. Find $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$. Use L'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If L'Hospital's Rule doesn't apply, explain why.

This limit has the form $\frac{0}{0}$.

$\mathbb{H} \Rightarrow \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$. This limit has the form $\frac{0}{0}$.

$\mathbb{H} \Rightarrow \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x}$. This limit has the form $\frac{0}{0}$.

$\mathbb{H} \Rightarrow \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = 2.$

Problem: **62**. **Find** $\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}}$. **Use L'Hospital's Rule where appropriate**. **If there is a more elementary method**, **consider using it**. **If L'Hospital's Rule doesn't apply**, **explain why**.

Let $y = (e^x + x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x} \ln(e^x + x)$, so...

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x+x)}{x}.$$ This limit has the form $\frac{\infty}{\infty}$.

$$\Rightarrow \mathbb{H} \Rightarrow \lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x+1)}{1} = \lim_{x\to\infty} \frac{e^x+1}{e^x+x}.$$ This limit has the form $\frac{\infty}{\infty}$.

$$\Rightarrow \mathbb{H} \Rightarrow \lim_{x\to\infty} \frac{e^x}{e^x+1}.$$ This limit has the form $\frac{\infty}{\infty}$.

$$\Rightarrow \mathbb{H} \Rightarrow \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1.$$

We just found $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \ln(e^x+x)^{\frac{1}{x}} = 1$, but we have NOT found $\lim_{x\to\infty} (e^x+x)^{\frac{1}{x}}$.

However, observe that: $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\left(\lim_{x\to\infty}\ln y\right)} = e^1 = e.$

Note that we were allowed to bring the limit into the exponential function $e^{\ln y}$. What justifies this? This is a result of the fact that $\ln y \to 1$ as $x \to \infty$, and e^x is continuous around $x = 1$ (see Theorem 8 in 2.5 regarding limits of continuous functions).

We could actually change the notation somewhat and see this as $\lim_{x\to\infty} e^{\ln y} = \lim_{x\to 1} e^x = e^1 = e.$

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Problem 65 **Find** $\lim_{x\to 0^+} (4x+1)^{\cot x}$. **Use L'Hospital's Rule where appropriate**. **If there is a more elementary method**, **consider using it**. **If L'Hospital's Rule doesn't apply**, **explain why**.

Observe that when you try to apply the limit, you get something of the form 1^∞ (recall that $\cot x = \frac{\cos x}{\sin x}$, and $\lim_{x\to 0^+} \frac{\cos x}{\sin x} = \frac{1}{0^+} = \infty$).

So, from the review we wish to use the Intermediate Powers method. The main purpose of the method is to get rid of the pesky expression in the exponent.

Let $y := (4x+1)^{\cot x}$.
Taking the natural log of both sides: $\ln y = \ln((4x+1)^{\cot x}) = \cot x \ln(4x+1)$.

Now that we have gotten rid of that exponent, we can proceed with attempting to take the limit.

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot x \cdot \ln(4x+1)$, which is of the form $\infty \cdot 0$.

So, we must now use the L'Hospital Intermediate Products method to try to solve this.

If we let $f = \cot x$ and $g = \ln(4x+1)$, we could rewrite this as $\lim_{x\to 0^+} \frac{f}{\frac{1}{g}}$ or $\lim_{x\to 0^+} \frac{g}{\frac{1}{f}}$.

The first one gives us: $\lim_{x\to 0^+} \frac{\cot x}{\ln^{-1}(4x+1)}$, and applying L'Hospital (taking the derivative of the numerator and denominator separately, NOT the quotient rule!), we have:

$$\lim_{x\to 0^+} \frac{-\csc^2 x}{-(\frac{4}{4x+1})\ln^{-2}(4x+1)} = \lim_{x\to 0^+} \frac{(4x+1)\ln^2(4x+1)}{4\sin^2 x}.$$

However, this looks more complicated than what we started with.

ProTip: With these Intermediate Products problems, if after taking the derivatives things do not get simpler, it's probably best to check out the other option (for us, $\lim_{x\to 0^+} \frac{g}{\frac{1}{f}}$), to see if that works out better.

In our case, that would be: $\lim_{x\to 0^+} \frac{g}{\frac{1}{f}} = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan x}$, and applying L'Hospital, we have:

$$\lim_{x\to 0^+} \frac{\frac{4}{4x+1}}{\sec^2 x} \to \frac{\frac{4}{1}}{\frac{1}{\cos^2(0)}} = 4.$$

But we are not done yet! We have found out that: $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot x \ln(4x+1) = 4$.

However, we were asked to evaluate: $\lim_{x\to 0^+} y = \lim_{x\to 0^+} (4x+1)^{\cot x}$.

So, to finish off this type of problem, we must notice that $y = e^{\ln y}$ is always true, and so:

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\left(\lim_{x\to 0^+} \ln y\right)} = e^4.$$

(We can do this trick with the limit because exponential functions are continuous everywhere, and therefore we are allowed to pull the limit into the exponent as we did in the 2nd equation above. See

Theorem 8 in 2.5 regarding limits of continuous functions)

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Now I will do a problem like $\lim_{x \to a} (f - g)$ that has the form "$\infty - \infty$." I will do one that involves factoring out a common factor, since the problems that involve merely finding a common denominator are easy. They are easy, because all you need to do is find the common denominator of f and g, combine f and g under the new denominator, and then proceed with L'Hospital as usual. However, the following problem does not start out as fractions, so we must instead try to factor out a common factor.

Problem 53. **Find** $\lim_{x \to \infty} (x - \ln x)$. **Use L'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If L'Hospital's Rule doesn't apply, explain why.**

The limit has the form $\infty - \infty$, and we will change the form to a product by factoring out an x.

$$= \lim_{x \to \infty} x\left(1 - \frac{\ln x}{x}\right) \qquad (*)$$

To determine what form $(*)$ has, we need to know what $\lim_{x \to \infty} \left(1 - \frac{\ln x}{x}\right)$ is.

Observe that $\lim_{x \to \infty} \frac{\ln x}{x}$ is of the form $\frac{\infty}{\infty}$, so we can calculate:

$$\lim_{x \to \infty} \frac{\ln x}{x} \Rightarrow \mathbb{H} \Rightarrow \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0.$$

So it must be that $\lim_{x \to \infty} \left(1 - \frac{\ln x}{x}\right) = \lim_{x \to \infty} 1 - \lim_{x \to \infty} \frac{\ln x}{x} = 1 - 0 = 1.$

So, going back to $(*)$, we have:

$$\lim_{x \to \infty} x\left(1 - \frac{\ln x}{x}\right) = \lim_{x \to \infty} x \cdot \lim_{x \to \infty} \left(1 - \frac{\ln x}{x}\right) = \infty \cdot 1 = \infty.$$