MATH 1271: Calculus I

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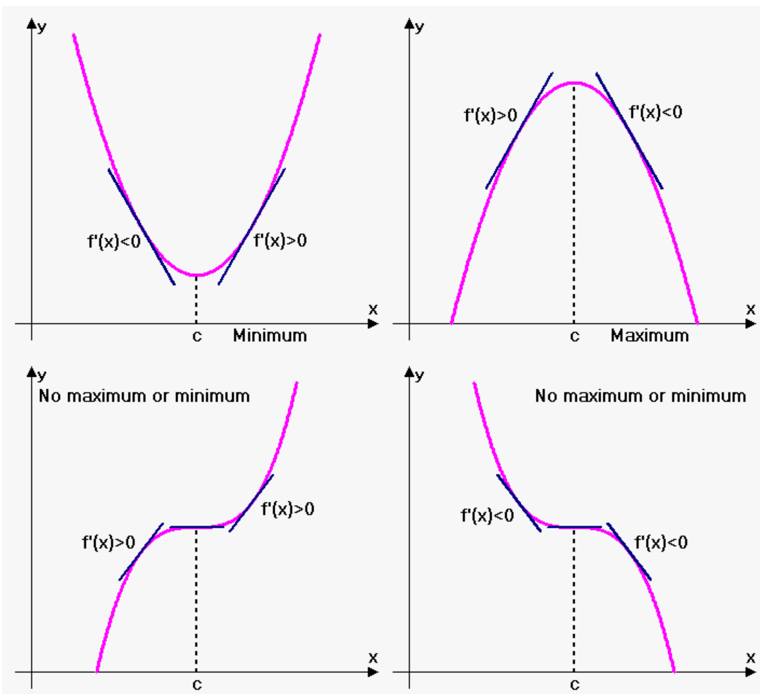
4.3 - How Derivatives Affect the Shape of a Graph

Review:

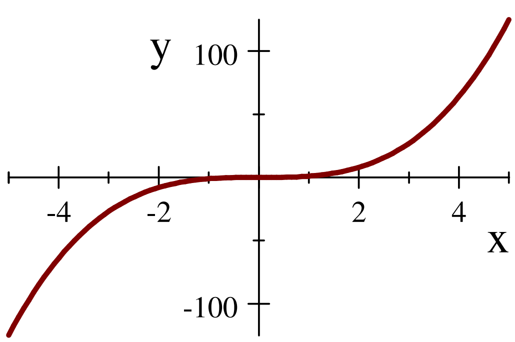


Increasing/Decreasing (**I/D Test for continuous functions**):

- Places in the function where it is increasing: $f'(x) > 0$.
- Places in the function where it is decreasing: $f'(x) < 0$.

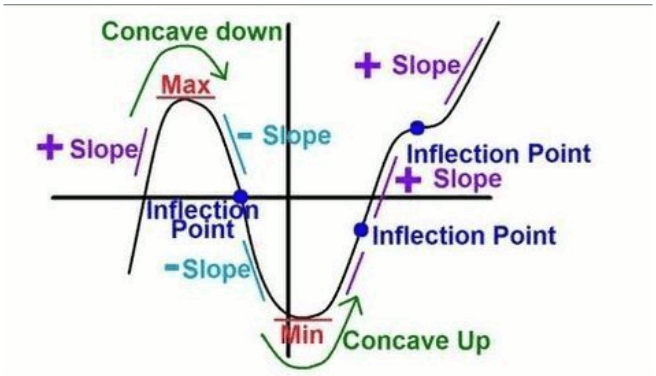
Note that if a function is increasing on (a,b) and also on (b,c), where b is a critical point $f'(b) = 0$, then the function is considered to be increasing on the entire interval (a,c).
The previous statement is also true if you change the words "increasing" all to "decreasing."

The most common example of this is that x^3 is considered to be increasing on $\mathbb{R}$, despite the fact that $f'(0) = 0$.



$f = x^3$

First Derivative Test
Assume c is a critical number of a continuous function f :
- If f' changes from positive to negative at c, then f has a local maximum there.
- If f' changes from negative to positive at c, then f has a local minimum there.
- If f' does not change sign at c, then f has no local maximum or minimum there.

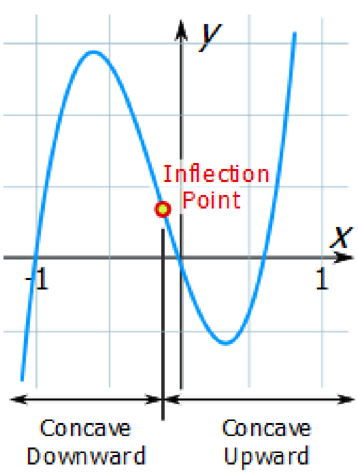


Another useful thing to know is that, for a continuous function, the sign of the derivative cannot change in between critical numbers.

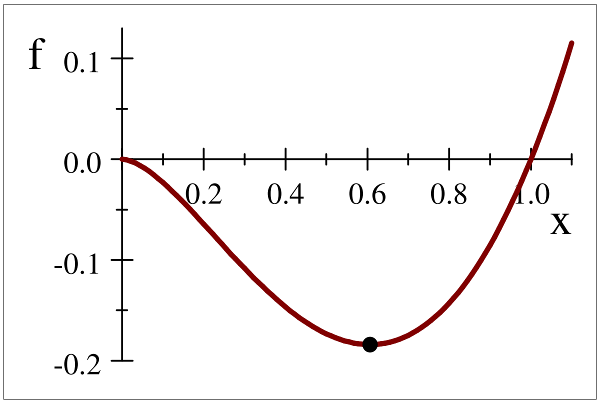
Concavity Test (**AKA Curvature**)
- If $f''(x) > 0$ for all x in some interval I, then the graph of f is concave upward (CU) there.
- If $f''(x) < 0$ for all x in some interval I, then the graph of f is concave downward (CD) there.

Second Derivative Test
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum there
  (critical point & positive curvature)
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum there.
  (critical point & negative curvature)

Inflection Point: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward (CU) to concave downward (CD), or if it changes from concave downward to concave upward.



Problem 16a. Find the intervals on which $f = x^2 \ln x$ is increasing or decreasing.



$f = x^2 \ln x$

$f'(x) = 2x \cdot \ln x + x^2(\frac{1}{x})$

$= 2x \ln x + x$

$= x(1 + 2\ln x) = 0.$ (where are the zeros?)

Note that the domain of f' is $(0, \infty)$. So $x = 0$ is not a zero of the derivative, and the sign of f' is determined solely by the zero of $1 + 2\ln x$.

$f'(x) = 0$ when $1 + 2\ln x = 0,$ $\ln x = -\frac{1}{2},$ $x = e^{-\frac{1}{2}}$ (≈ 0.61).

So checking a point to the left of $e^{-\frac{1}{2}}$, we have $f'(0.5) = 1 + 2\ln(0.5) \approx -0.386$. So we conclude that $f'(x) < 0$ when $0 < x < e^{-\frac{1}{2}}$.

Similarly, calculating $f'(1) = 1$, we see that $f'(x) > 0$ when $e^{-\frac{1}{2}} < x$.

So f is decreasing on $\left(0, e^{-\frac{1}{2}}\right)$ and f is increasing on $\left(e^{-\frac{1}{2}}, \infty\right)$.

16b. **Find the local maximum and minimum values of** $f = x^2 \ln x$.

f changes from decreasing to increasing at $x = e^{-\frac{1}{2}}$.

Thus, $f\left(e^{-\frac{1}{2}}\right) = \left(e^{-\frac{1}{2}}\right)^2 \ln\left(e^{-\frac{1}{2}}\right) = e^{-1}(-\frac{1}{2}) = -\frac{1}{2e}$ (≈ -0.18) is a local minimum value.

16c. **Find the intervals of concavity and the inflection points**.

Recall: $f'(x) = x(1 + 2\ln x)$.

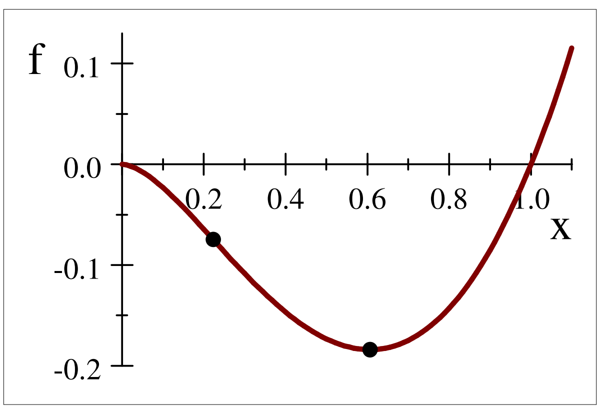
$$\Rightarrow f''(x) = 1 \cdot (1 + 2\ln x) + x(\tfrac{2}{x}) = 3 + 2\ln x.$$

$$f''(x) > 0 \text{ when } 3 + 2\ln x > 0 \quad \Rightarrow \quad \ln x > -\tfrac{3}{2} \quad \Rightarrow \quad x > e^{-\frac{3}{2}} \quad (\approx 0.22).$$

Thus, f is concave downward on $\left(0, e^{-\frac{3}{2}}\right)$ and f is concave upward on $\left(e^{-\frac{3}{2}}, \infty\right)$. What is the inflection point?

$$f\left(e^{-\frac{3}{2}}\right) = \left(e^{-\frac{3}{2}}\right)^2 \ln e^{-\frac{3}{2}} = e^{-3}(-\tfrac{3}{2}) = -\tfrac{3}{2e^3} \quad (\approx -0.07).$$

There is a point of inflection at $(x,y) = \left(e^{-\frac{3}{2}},\ f\left(e^{-\frac{3}{2}}\right)\right) = \left(e^{-\frac{3}{2}},\ -\frac{3}{2e^3}\right)$.



$f = x^2 \ln x$

Problem 44a. Find the intervals of increase or decrease for $S(x) = x - \sin x$, on $0 \le x \le 4\pi$.

$S'(x) = 1 - \cos x$.

So, $S'(x) = 0$ when $\cos x = 1$

$\Rightarrow \quad x = 0,\ 2\pi,\ \text{and } 4\pi.$

The function is increasing ($S'(x) > 0$) when $\cos x < 1$, which is true for all x except integer multiples of 2π, so S is increasing on $(0, 4\pi)$.

Recall: "if a function is increasing on (a,b) and also on (b,c), where b is a critical point $f'(b) = 0$, then the function is considered to be increasing on the entire interval (a,c)."

b. Find the local maximum and minimum values.

There is no local maximum or minimum.

Recall: "If S' does not change sign at c, then S has no local maximum or minimum there."

c. Find the intervals of concavity and the inflection points.

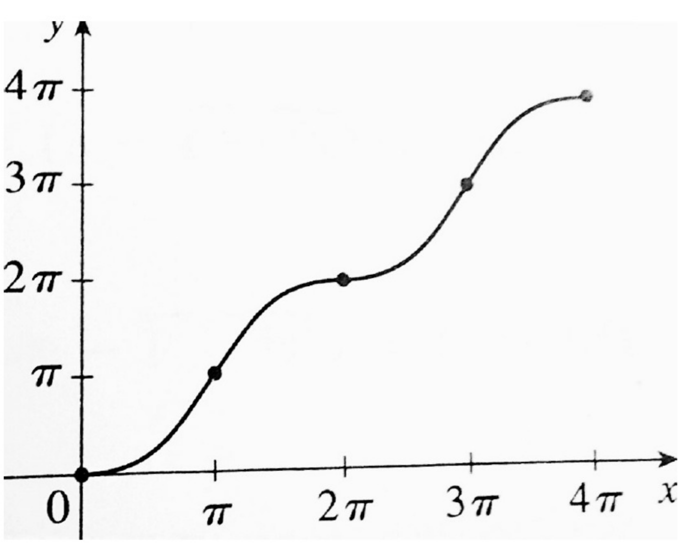
$S''(x) = \sin x.$

So, $S''(x) > 0$ if $0 < x < \pi$ or $2\pi < x < 3\pi$, and $S''(x) < 0$ if $\pi < x < 2\pi$ or $3\pi < x < 4\pi$.

So S is CU on $(0, \pi)$ and $(2\pi, 3\pi)$, and S is *CD* on $(\pi,\ 2\pi)$ and $(3\pi,\ 4\pi)$.

Therefore there are inflection points at $(\pi, S(\pi)) = (\pi, \pi),\ (2\pi, 2\pi),$ and $(3\pi, 3\pi)$.

d. Use the information from the previous parts to sketch the graph. Check your work with a graphing device if you have one.



Problem 48. Find the following qualitative features for $f(x) = \frac{e^x}{1-e^x}$.

a. Find the domain, as well as the vertical and horizontal asymptotes (VA &HA).

Has domain $\{x \mid 1 - e^x \neq 0\} = \{x \mid e^x \neq 1\} = \{x \mid x \neq 0\}$.

$\lim_{x\to 0^+} \frac{e^x}{1-e^x} = -\infty$ and $\lim_{x\to 0^-} \frac{e^x}{1-e^x} = \infty$, so $x = 0$ is a VA.

$\lim_{x\to\infty} \frac{e^x}{1-e^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{(1-e^x)}{e^x}} = \lim_{x\to\infty} \frac{1}{\frac{1}{e^x}-1} = \frac{1}{0-1} = -1$, so $y = -1$ is a HA.

$\lim_{x\to-\infty} \frac{e^x}{1-e^x} = \frac{0}{1-0} = 0$, so $y = 0$ is a HA.

b. Find the intervals of increase or decrease.

$$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \frac{e^x[(1-e^x)+e^x]}{(1-e^x)^2} = \frac{e^x}{(1-e^x)^2}.$$

So, $f'(x) > 0$ for $x \neq 0$, and f is increasing on $(-\infty, 0)$ and $(0, \infty)$.

c. Find the local maximum and minimum values.

Since the function is always increasing in its domain, there is no local maximum or minimum.

d. Find the intervals of concavity and the inflection points.

$$f''(x) = \frac{(1-e^x)^2 e^x - e^x \cdot 2(1-e^x)(-e^x)}{\left[(1-e^x)^2\right]^2} = \frac{(1-e^x)e^x[(1-e^x)+2e^x]}{(1-e^x)^4} = \frac{e^x(e^x+1)}{(1-e^x)^3}.$$

Since the numerator is always positive, $f''(x) > 0$ when $(1-e^x)^3 > 0$, which implies $e^x < 1$ or $x < 0$.

Therefore, $f''(x) < 0$ when $x > 0$.

So f is CU on $(-\infty, 0)$ and f is CD on $(0, \infty)$.

There can be no inflection point, as the function never transitions from CU↔CD within its domain.

e. **Use the information from the previous parts to sketch the graph**.