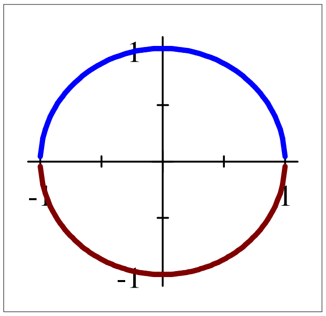
MATH 1271: Calculus I

Discussion Instructor: Jodin Morey
moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

3.5 - Implicit Differentiation

Review



Some algebraic expressions cannot be expressed "**explicitly**," which means it cannot be expressed as a **function** of the form $y = f(x)$.

For example, the equation for a unit circle: $x^2 + y^2 = 1$. In this case, the closest we can get is to split it up into $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ (the upper and lower half of the unit circle as above). So how do you find the derivative of these "implicit" types of equations?

Implicit Differentiation: First recall that when we take a derivative, we do so with respect to a particular variable. In the case of the circle above, the more traditional choice would be to take the derivative of $x^2 + y^2 = 1$ with respect to the horizontal axis x (instead of y). So we take the derivative $\frac{d}{dx}$ of both sides of the equation $x^2 + y^2 = 1$. When we do this, we keep in mind the chain rule, and treat any other variable (other than x) that we run across (like y) as a function of x. In other words, we think of y as $y(x)$.

For example:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1),$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0,$$

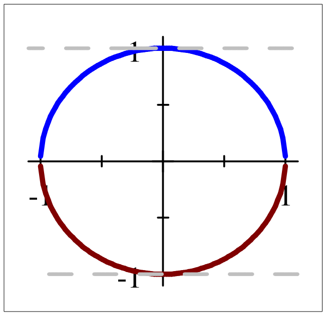
$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0,$$ (using the chain rule, and treating y as a function of x)

$$2x(1) + 2y\frac{dy}{dx} = 0,$$ (now let's solve for $\frac{dy}{dx}$)

$$2y\frac{dy}{dx} = -2x,$$

$$\frac{dy}{dx} = -\frac{x}{y} \text{ or } y' = -\frac{x}{y}.$$

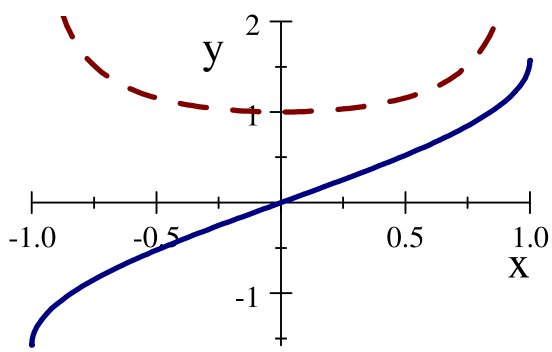
So we can plug in any point (a,b) from the unit circle, and the derivative $y' = -\frac{a}{b}$ tells you the slope of the circle at the point (a,b). For example, the slope at $a = 0$ of the circle is $y' = -\frac{0}{b} = 0$, as expected (for $b = \pm 1$).



Derivatives of Inverse Trigonometric Functions:

$\frac{d}{dx}(\tan^{-1}) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

One might wonder how frequently the above table is needed. They are particularly useful when performing integration, which we will learn about later in the course.



$\sin^{-1}x$ (solid) and $\frac{1}{\sqrt{1-x^2}}$ (dashed)

Problem 7. Find the $\frac{d}{dx}$ derivative of $x^2 + xy - y^2 = 4$ by implicit differentiation.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$2x + \left(\frac{dx}{dx}y + x\frac{dy}{dx}\right) - (2y)y' = 0$$

$$2x + [1 \cdot y + xy'] - 2yy' = 0 \qquad \text{(now we solve for } y'\text{)}$$

$$xy' - 2yy' = -2x - y$$

$$y'(x - 2y) = -2x - y$$

$$y' = \frac{-2x-y}{x-2y} = \frac{-(2x+y)}{-(2y-x)} = \frac{2x+y}{2y-x}.$$

Problem 17. Find y' by implicit differentiation: $\tan^{-1}(x^2y) = x + xy^2$

$$\frac{d}{dx}\tan^{-1}(x^2y) = \frac{d}{dx}(x+xy^2)$$

$$\frac{1}{1+(x^2y)^2}(2xy + x^2y') = 1 + (1)y^2 + x(2yy')$$

(inverse trig derivative, chain rule, product rule, and power rule!)

$$\frac{2xy}{1+x^4y^2} + \frac{x^2}{1+x^4y^2}y' = 1 + y^2 + 2xyy' \qquad \text{(now we solve for } y'\text{)}$$

$$\frac{x^2}{1+x^4y^2}y' - 2xyy' = 1 + y^2 - \frac{2xy}{1+x^4y^2}$$

$$y'\left(\frac{x^2}{1+x^4y^2} - 2xy\right) = 1 + y^2 - \frac{2xy}{1+x^4y^2}$$

$$y' = \frac{1+y^2-\frac{2xy}{1+x^4y^2}}{\frac{x^2}{1+x^4y^2}-2xy}.$$

Then, by multiplying by $\frac{1+x^4y^2}{1+x^4y^2}$, we get $y' = \frac{(1+y^2)(1+x^4y^2)-2xy}{x^2-2xy(1+x^4y^2)} = \frac{x^4y^4+x^4y^2+y^2-2xy+1}{x^2-2xy-2x^5y^3}$.

Problem 27. Use implicit differentiation to find an equation of the tangent line to $x^2 + xy + y^2 = 3$ (an ellipse) at the point $(1,1)$.

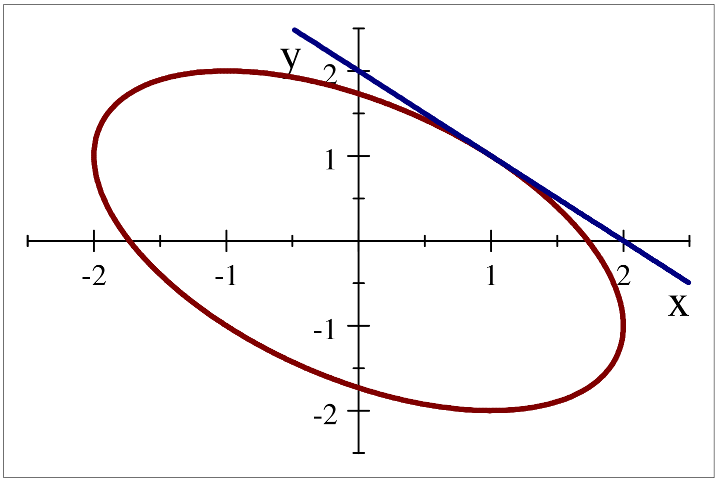
$$2x + (1 \cdot y + xy') + 2yy' = 0$$

$$xy' + 2yy' = -2x - y$$

$$y'(x+2y) = -2x - y$$

$y' = \frac{-2x-y}{x+2y}.$

When $x = 1$ and $y = 1$, we have slope: $y' = \frac{-2-1}{1+2\cdot 1} = \frac{-3}{3} = -1$, so an equation of the tangent line is $y - 1 = -1(x - 1)$ or $y = -x + 2$.



$x^2 + xy + y^2 = 3$ and $y = -x + 2$

Problem 37. Find y'' by implicit differentiation of $x^3 + y^3 = 1$.

$3x^2 + 3y^2y' = 0$

$y' = -\frac{x^2}{y^2}$

$y'' = -\frac{(2x)y^2-(x^2)2yy'}{(y^2)^2}$

(the expectation is that you eliminate any derivatives from the right hand side of the equation)

$$= -\frac{2xy^2-2x^2y\left(\frac{-x^2}{y^2}\right)}{y^4}$$

$$= -\frac{2xy^2+2x^4\left(\frac{1}{y}\right)}{y^4}$$

$$= -\frac{2xy^3+2x^4}{y^5}$$

$= -\frac{2x(y^3+x^3)}{y^5}$

(observe that this can be simplified further with information from above)

$= -\frac{2x}{y^5}$. Since x and y must satisfy the original equation, $x^3 + y^3 = 1$.

Problem 44. Show by implicit differentiation that the tangent to the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is: $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.

$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

$\frac{2yy'}{b^2} = -\frac{2x}{b^2}$

$y' = -\frac{b^2 x}{a^2 y}$

An equation of the tangent line at (x_0, y_0) is $y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0)$.

Let's try to rearrange this to look a little more like our original equation, multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0 x}{a^2} - \frac{x_0^2}{a^2}$.

$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}$.

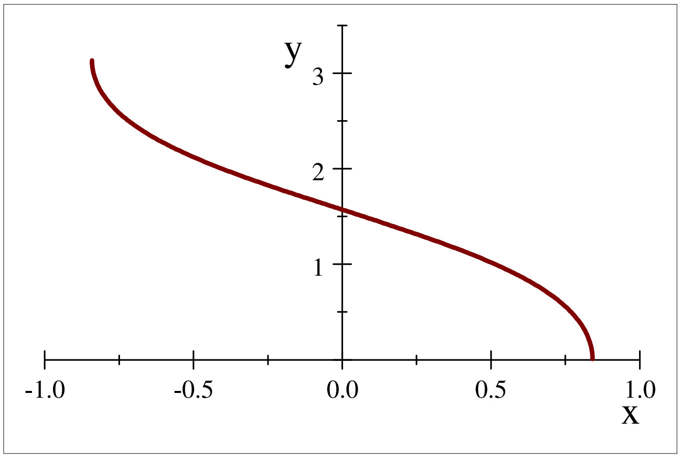
Since (x_0, y_0) lies on the ellipse, we have $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$.

Problem 58. Find the derivative of $y = \cos^{-1}(\sin^{-1} t)$. Simplify where possible.

$y' = -\frac{1}{\sqrt{1-(\sin^{-1}t)^2}} \cdot \left(\frac{d}{dt} \sin^{-1} t\right)$

$$= -\frac{1}{\sqrt{1-\sin^{-2} t}} \cdot \frac{1}{\sqrt{1-t^2}}.$$

Graph of y



Graph of y'