MATH 1271: Calculus I

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3.4 - Chain Rule

Review

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) := f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = \left(f(g)\right)' = f'(g(x)) \cdot g'(x)$.

For example let $f := \sin(x)$ and $g := x^2$. Then $F = f(g) = \sin(x^2)$ and
$F' = f'(g(x)) \cdot g'(x) = \cos(x^2) \cdot (2x)$.

Derivative in Leibniz notation: $\frac{dF}{dx} = \frac{df(g(x))}{dx} = \frac{df(g(x))}{dg}\frac{dg}{dx}$.

Alternate notation: $F = f \circ g$.

Power Rule Combined with the Chain Rule: If n is any real number and $u = g(x)$ is differentiable, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$. Alternative notation: $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$.

Problem 17. Find the derivative of $f(x) = (x^2+x+1)^5(2x-3)^4$.

We think of $g := x^2+x+1$ and $h := 2x-3$, so that $f = g^5h^4$.

By product and chain rule, we have:

$$f' = (g^5)'h^4 + g^5(h^4)' = (5g^4 \cdot g')h^4 + g^5(4h^3 \cdot h')$$

$$= 5(x^2+x+1)^4(2x+1)(2x-3)^4 + (x^2+x+1)^5 \cdot 4(2x-3)^3 \cdot 2$$

$$= 5(x^2+x+1)^4(2x+1)(2x-3)^4 + 8(x^2+x+1)^5(2x-3)^3 \qquad \text{(you could stop here)}$$

$$= (2x-3)^3(x^2+x+1)^4[5(2x-3)(2x+1) + 8(x^2+x+1)]$$

$$= (2x-3)^3(x^2+x+1)^4(20x^2-20-15+8x^2+8x+8)$$

$= (2x-3)^3(x^2+x+1)^4(28x^2-12x-7).$

Problem 31. Find the derivative of $y = \sin(\tan 2x)$.

Let $u = \tan 2x$. Therefore, $y = \sin u$. Note: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

$\frac{dy}{dx} = \cos u \cdot \frac{d}{dx}(\tan 2x)$

$= \cos(\tan 2x) \cdot \sec^2(2x) \cdot \frac{d}{dx}(2x)$

$= 2\cos(\tan 2x)\sec^2(2x).$

So we actually had $y = f(u(v))$, where $f = \sin(x)$, $u = \tan(x)$, and $v = 2x$. (double chain rule!)

Problem 39. Find the derivative of $f(t) = \tan(e^t) + e^{\tan t}$.

$y' = \sec^2(e^t) \cdot \frac{d}{dt}(e^t) + e^{\tan t} \cdot \frac{d}{dt}(\tan t)$

$= \sec^2(e^t) \cdot e^t + e^{\tan t} \cdot \sec^2(t)$

$= e^t \sec^2(e^t) + e^{\tan t}\sec^2(t)$

Problem 47. Find y' and y'' of $y = \cos(x^2)$.

$y' = -\sin(x^2) \cdot 2x = -2x\sin(x^2)$

$y'' = (-2)\sin(x^2) - 2x\cos(x^2) \cdot 2x$ (product and chain rule!)

$= -2\sin(x^2) - 4x^2\cos(x^2).$

Problem 53. Find an equation of the tangent line to $y = \sin(\sin x)$ at the point $(\pi, 0)$.

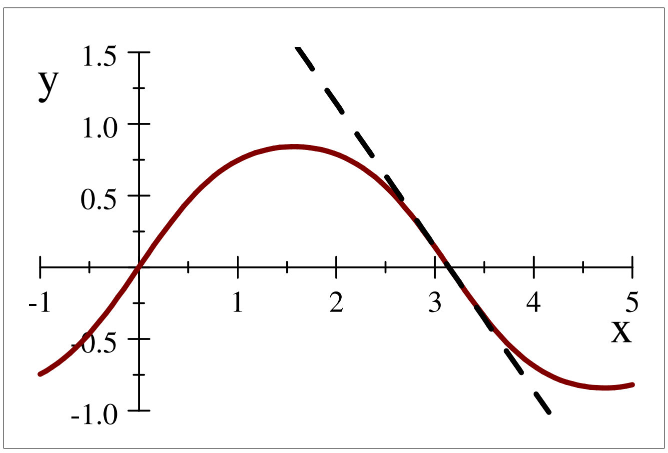
$y' = \cos(\sin x) \cdot \cos x$.

At $(\pi, 0)$, $y' = \cos(\sin \pi) \cdot \cos \pi$

$= \cos(0) \cdot (-1) = 1(-1) = -1$.

And an equation of the tangent line is

$y - 0 = -1(x - \pi)$, or $y = -x + \pi$.



$\sin(\sin x)$ and $y = -x + \pi$

Problem 73. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

$F' = f'(3f(4f)) \cdot \frac{d}{dx}(3f(4f)) = f'(3f(4f)) \cdot 3f'(4f) \cdot \frac{d}{dx}(4f)$

$= f'(3f(4f)) \cdot 3f'(4f) \cdot 4f'$, so...

$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0)$

$= f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2$

$$= 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96.$$

Problem 78. Find the $1,000th$ derivative of $f(x) = xe^{-x}$.

$$f' = e^{-x} - xe^{-x} = (1-x)e^{-x},$$

$$f'' = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}.$$

Similarly, $f''' = (3-x)e^{-x}$,

$$f^{(4)} = (x-4)e^{-x}, \ldots, f^{(1,000)} = (x-1000)e^{-x}.$$