MATH 1271: Calculus I

Discussion Instructor: Jodin Morey
moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

3.2 - Product and Quotient Rules

Review

Product Rule: If f and g are both differentiable (on some interval),
then $\frac{d}{dx}[f \bullet g] = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$ (on that same interval).
Equivalently: $(f \bullet g)' = f\,'g + fg'$ or equivalently $gf\,' + fg'$.

Quotient Rule: If f and g are differentiable, then $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$.
Equivalently: $\left(\frac{f}{g}\right)' = \frac{gf\,'-fg'}{g^2}$.

Remember the saying:
"low di-hi minus hi di-low, square the bottom, and away we go!"

Problem 6. Differentiate $y = \frac{e^x}{1-e^x}$.

So, $y' = \frac{e^x(1-e^x)-e^x(-e^x)}{(1-e^x)^2}$

$= \frac{e^x-e^{2x}+e^{2x}}{(1-e^x)^2}$

$= \frac{e^x}{(1-e^x)^2}$.

Problem 11. Differentiate $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y+5y^3)$

$= (y^{-2} - 3y^{-4})(y+5y^3)$

$F\,' = (y^{-2} - 3y^{-4})'(y+5y^3) + (y^{-2} - 3y^{-4})(y+5y^3)'$

$= (-2y^{-3} + 12y^{-5})(y+5y^3) + (y^{-2} - 3y^{-4})(1+15y^2)$

(now you have completed the differentiation, although it is usually expected that you try to simplify the result)

$$= (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2}) + (y^{-2} + 15 - 3y^{-4} - 45y^{-2})$$

$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + \frac{14}{y^2} + \frac{9}{y^4}.$$

Problem. Differentiate $y = |x|$.

$$y' = \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x} \cdot \frac{|x+\Delta x| + |x|}{|x+\Delta x| + |x|}$$

$$= \lim_{\Delta x \to 0} \frac{|x+\Delta x|^2 - |x|^2}{\Delta x(|x+\Delta x| + |x|)}$$

The Key Insight: Observe that for $x > 0$, we have $|x|^2 = x^2$, and for $x < 0$, we have $|x|^2 = x^2$, and similarly for $x + \Delta x > 0$ or $x + \Delta x < 0$. Therefore, our difference-of-squares trick will allow us to eliminate the absolute value signs, and simplify the numerator in preparation for a cancellation between the numerator and denominator.

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x(|x+\Delta x| + |x|)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x(|x+\Delta x| + |x|)} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x(|x+\Delta x| + |x|)}$$

$$= \lim_{\Delta x \to 0} \frac{2x + \Delta x}{|x+\Delta x| + |x|} = \frac{x}{|x|}.$$

Observe that this derivative does not exist for $x = 0$, but is ± 1 for all other x.

Problem 25. Differentiate $f(x) = \frac{x}{x + \frac{c}{x}}$ where c is a constant.

First simplify by eliminating the double denominator: $f(x) = \frac{x^2}{x^2+c}$.

$$f' = \frac{ba'-ab'}{b^2}$$

$$= \frac{2x(x^2+c)-x^2(2x)}{(x^2+c)^2}$$

$$= \frac{2cx}{(x^2+c)^2}.$$

Problem 29. Find $f'(x)$ and $f''(x)$ for $f(x) = \frac{x^2}{1+2x}$.

$$f' = \frac{(2x)(1+2x)-x^2(2)}{(1+2x)^2} = \frac{2x+4x^2-2x^2}{(1+2x)^2} = \frac{2x^2+2x}{(1+2x)^2}$$

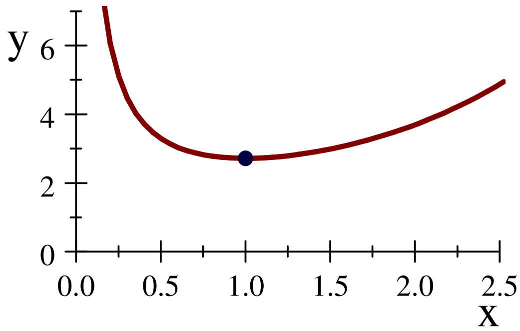
$$f'' = \frac{(4x+2)(1+2x)^2-(2x^2+2x)(4x^2+4x+1)'}{\left[(1+2x)^2\right]^2}$$

$$= \frac{(4x+2)(1+2x)^2-(2x^2+2x)(4+8x)}{(1+2x)^4}$$

$$= \frac{(4x+2)(1+2x)^2-4(2x^2+2x)(1+2x)}{(1+2x)^4}$$

$$= \frac{(4x+2)(1+2x)-4(2x^2+2x)}{(1+2x)^3} = \frac{(4x+2+8x^2+4x)-(8x^2+8x)}{(1+2x)^3} = \frac{2}{(1+2x)^3}.$$

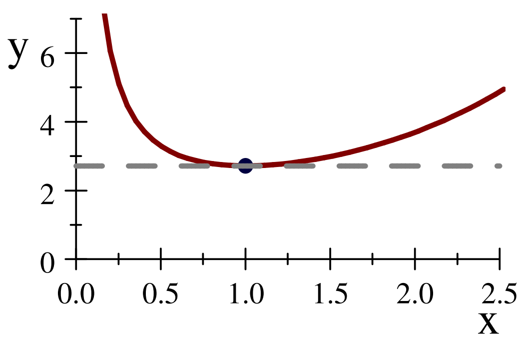
Problem 32. Find an equation of the tangent line to $y = \frac{e^x}{x}$ at the point $(1, e)$.



$y' = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$.

At $x = 1$, observe that $y' = 0$.

And at the point $(1, e)$, an equation of the tangent line is $y - e = 0(x - 1)$, or $y = e$.

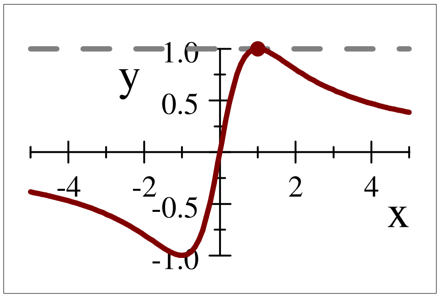


Problem 34. Find the equations of the tangent line and the normal line for $y = \frac{2x}{x^2+1}$ at the point $(1, 1)$.

$y' = \frac{2 \cdot (x^2+1) - 2x \cdot (2x)}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2}$.

At $x = 1$, $y' = 0$.

And at $(1, 1)$, an equation of the tangent line is $y - 1 = 0(x - 1)$, or $y = 1$.



The slope of the normal line is $\frac{1}{0}$ $=$ undefined, so the normal line at $(1,1)$ must be vertical, so the equation of the normal line is $x = 1$.

Problem 52. If f is a differentiable function, find an expression for the derivative of each of the following functions.

a. $y = x^2 f(x)$

$$y' = 2x \cdot f(x) + x^2 \cdot f'(x).$$

b. $y = \frac{f(x)}{x^2}$

$$y' = \frac{f'(x) \cdot x^2 - f(x) \cdot 2x}{(x^2)^2} = \frac{x f'(x) - 2f(x)}{x^3}.$$

c. $y = \frac{x^2}{f(x)}$

$$y' = \frac{(2x)f(x) - x^2 f'(x)}{[f(x)]^2}.$$