MATH 1271: Calculus I

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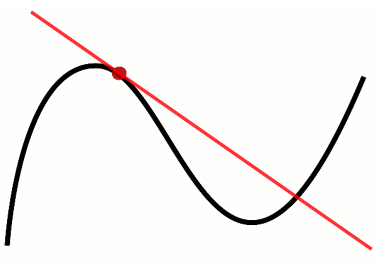
2.7 - Derivatives and Rates of Change

Review



Tangent: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ provided that this limit exists.

Or equivalently, $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Derivative of a Function: The **derivative of a function** f **at a number** a, denoted by $f'(a)$ is $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, **if this limit exists in** $(-\infty, \infty)$!

Velocity: If x is a position function, then at a we have the velocity:
$v(a) = x'(a) = \lim_{h \to 0} \frac{x(a+h)-x(a)}{h}$.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Instantaneous Rate of Change: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Problem 9a. Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

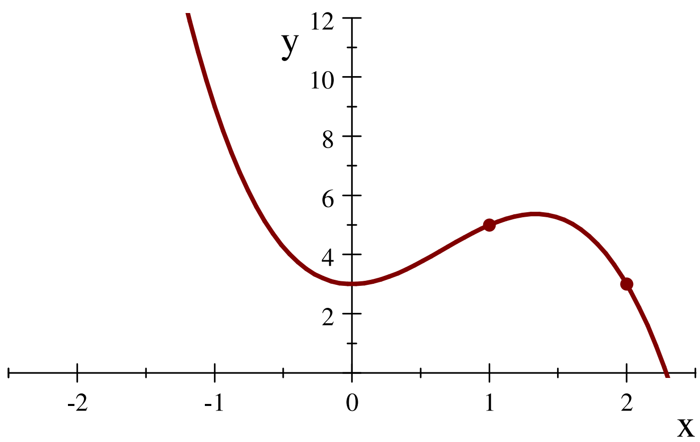
$$m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left(3+4(a+h)^2-2(a+h)^3\right)-(3+4a^2-2a^3)}{h}$$

$$= \lim_{h \to 0} \frac{3+4(a^2+2ah+h^2)-2(a^3+3a^2h+3ah^2+h^3)-3-4a^2+2a^3}{h} = \lim_{h \to 0} \frac{3+4a^2+8ah+4h^2-2a^3-6a^2h-6ah^2-2h^3-3-4a^2+2a^3}{h}$$

$$= \lim_{h \to 0} \frac{8ah+4h^2-6a^2h-6ah^2-2h^3}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{h(8a+4h-6a^2-6ah-2h^2)}{h} = \lim_{h \to 0} (8a+4h-6a^2-6ah-2h^2) = 8a-6a^2.$$



Problem 9b. Find equations of the tangent lines at the points $(1,5)$ and $(2,3)$.

At $(1,5)$:

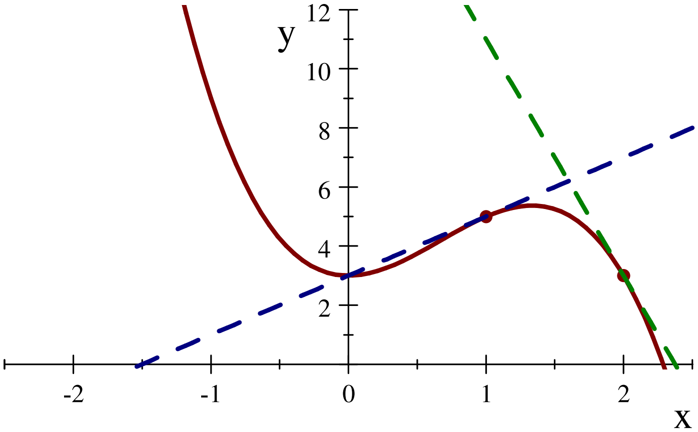
$m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is...

$y - 5 = 2(x-1) \Leftrightarrow y = 2x+3.$

At $(2,3)$:

$m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is…

$y - 3 = -8(x-2) \Leftrightarrow y = -8x+19.$



Problem 15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $x = \frac{1}{t^2}$, where t is measured in seconds. Find the velocity of the particle at time $t = a$.

$$v(a) = \lim_{h \to 0} \frac{x(a+h) - x(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h}$$

$$= \lim_{h \to 0} \frac{a^2 - (a+h)^2}{h \cdot a^2 (a+h)^2}$$ (by multiplying the numerator and denominator by $a^2(a+h)^2$)

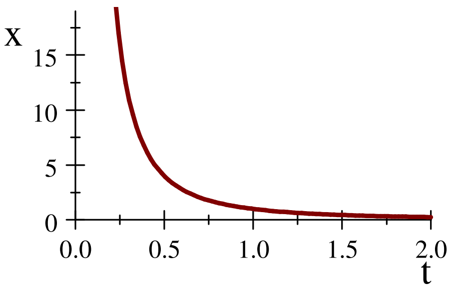
$$= \lim_{h \to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{a^2 h (a+h)^2}$$

$$= \lim_{h \to 0} \frac{-2ah - h^2}{ha^2 (a+h)^2}$$

$$= \lim_{h \to 0} \frac{-2a - h}{a^2 (a+h)^2}$$

$$= -\frac{2a}{a^2 \cdot a^2} = -\frac{2}{a^3} \frac{meters}{second}.$$

So, $v(1) = -2 \frac{m}{s}$, $v(2) = -\frac{1}{4} \frac{m}{s}$, and $v(3) = -\frac{2}{27} \frac{m}{s}$.



$\frac{1}{t^2}$ $-\frac{2}{t^3}$

Rolling down a hill?

Problem 31. Find $f'(a)$ of $f(x) = \sqrt{1-2x}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{1-2(a+h)}-\sqrt{1-2a}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-2(a+h)}-\sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)}+\sqrt{1-2a}}{\sqrt{1-2(a+h)}+\sqrt{1-2a}} = \lim_{h\to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2-\left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)}+\sqrt{1-2a}\right)}$$

$$= \lim_{h\to 0} \frac{(1-2a-2h)-(1-2a)}{h\left(\sqrt{1-2(a+h)}+\sqrt{1-2a}\right)}$$

$$= \lim_{h\to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)}+\sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{-2}{\sqrt{1-2(a+h)}+\sqrt{1-2a}}$$

$$= \frac{-2}{\sqrt{1-2a}+\sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}.$$

Problem 37. The limit ($\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h}$) represents the derivative of many functions f at some numbers a.

State such an f and a:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$f = \cos x$ and $a = \pi$ or …

$f = \cos(x+\pi)$ and $a = 0,$ or ...

$f = \cos(x-2k\pi)$ and $a = \pi(2k+1)$.