MATH 1271: Calculus I

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Challenge yourself during homework.
(**don't immediately look up the solution**!)
Calculus is not a spectator sport.

2.6 - Limits at Infinity; Horizontal Asymptotes

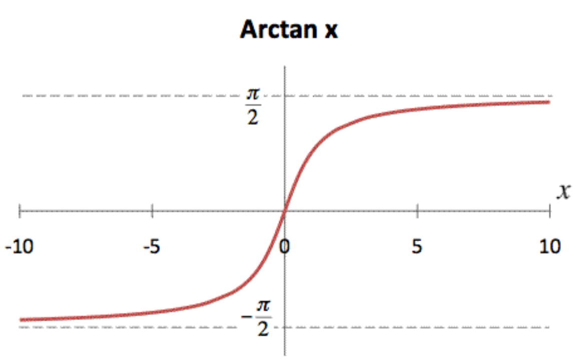
Review

Definition of Asymptotic Limit: Let f be a function defined on some interval (a,∞). Then, $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

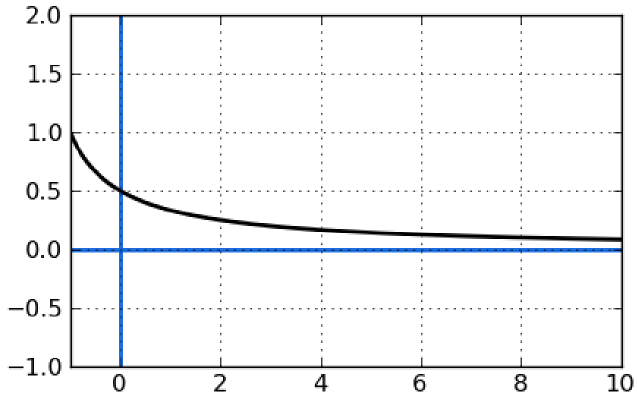
Similarly, Let f be a function defined on some interval $(-\infty,a)$. Then, $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently negative.

Horizontal Asymptote: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Observe that: $\lim_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2}$ and $\lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2}$.

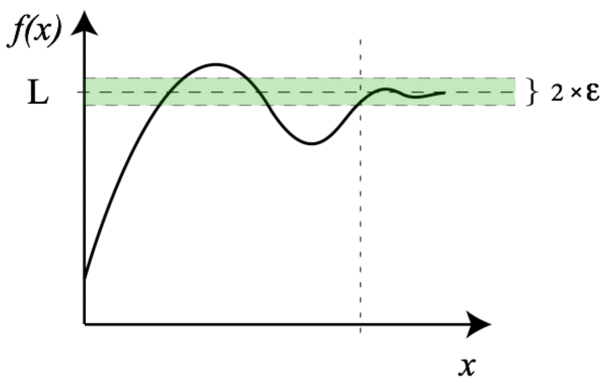


Theorem: Assuming $r > 0$ is a rational number: $\lim_{x\to\infty} \frac{1}{x^r} = 0$. Also, if x^r is defined for all x (which would not be true for $r = \frac{1}{2}$ for example), we have $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.



$\frac{1}{x^r},\ r > 0$

Precise Definition of Asymptotic Limit: Let f be a function defined on some interval (a, ∞). Then $\lim_{x \to \infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number of N such that if $x > N$, then $|f(x) - L| < \varepsilon$.

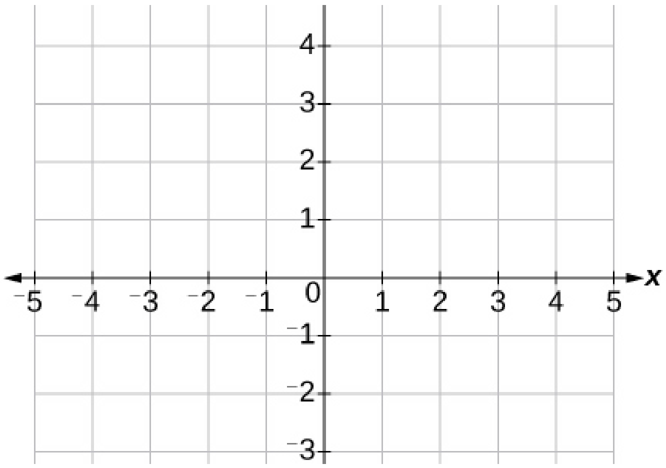


Infinity as a Limit: Let f be a function defined on some interval (a, ∞). Then, $\lim_{x \to \infty} f(x) = \infty$ means that for every positive number M, there is a corresponding positive number N such that if $x > N$, then $f(x) > M$.

Problem 9. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$f(0) = 3,\qquad \lim_{x \to 0^-} f(x) = 4,\qquad \lim_{x \to 0^+} f(x) = 2,$

$\lim_{x \to -\infty} f(x) = -\infty,\qquad \lim_{x \to 4^-} f(x) = -\infty,\qquad \lim_{x \to 4^+} f(x) = \infty.$



Problem 14. Evaluate the limit and **justify each step** by indicating the appropriate properties of limits.

$$\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}}$$

$$=\sqrt{\lim_{x\to\infty}\frac{12x^3-5x+2}{1+4x^2+3x^3}}$$ [Limit Law 11 (radicand > 0? We shall see.)]

$$=\sqrt{\lim_{x\to\infty}\frac{12-\frac{5}{x^2}+\frac{2}{x^3}}{\frac{1}{x^3}+\frac{4}{x}+3}}$$ [divide numerator/denominator by x^3]

$$=\sqrt{\frac{\lim_{x\to\infty}\left(12-\frac{5}{x^2}+\frac{2}{x^3}\right)}{\lim_{x\to\infty}\left(\frac{1}{x^3}+\frac{4}{x}+3\right)}}$$ [Limit Law 5, (limit of denominator = 0? We shall see.)]

$$=\sqrt{\frac{\lim_{x\to\infty}(12)-\lim_{x\to\infty}\left(\frac{5}{x^2}\right)+\lim_{x\to\infty}\left(\frac{2}{x^3}\right)}{\lim_{x\to\infty}\left(\frac{1}{x^3}\right)+\lim_{x\to\infty}\left(\frac{4}{x}\right)+\lim_{x\to\infty}(3)}}$$ [Limit Laws 1 and 2]

$$=\sqrt{\frac{12-5\lim_{x\to\infty}\left(\frac{1}{x^2}\right)+2\lim_{x\to\infty}\left(\frac{1}{x^3}\right)}{\lim_{x\to\infty}\left(\frac{1}{x^3}\right)+4\lim_{x\to\infty}\left(\frac{1}{x}\right)+3}}$$ [Limit Laws 3 and 7]

$$=\sqrt{\frac{12-5(0)+2(0)}{0+4(0)+3}}$$ [Theorem 5 of Section 2.6]

(Here we see that our first and 3rd step above were justified.)

$$= \sqrt{\frac{12}{3}} = \sqrt{4} = 2.$$

Problem 21. Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$$

$$= \lim_{x\to\infty} \frac{\frac{(2x^2+1)^2}{x^4}}{\frac{(x-1)^2(x^2+x)}{x^4}}$$

$$= \lim_{x\to\infty} \frac{\left[\frac{2x^2+1}{x^2}\right]^2}{\left[\frac{x^2-2x+1}{x^2}\right]\left[\frac{x^2+x}{x^2}\right]}$$

$$= \lim_{x\to\infty} \frac{\left(2+\frac{1}{x^2}\right)^2}{\left(1-\frac{2}{x}+\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)}$$

$$= \frac{(2+0)^2}{(1-0+0)(1+0)} = 4.$$

Problem 23. Find the limit or show that it does not exist: $\lim_{x\to\infty} \frac{\sqrt{9x^6-6}}{x^3+1}$.

$$= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^6-6}}{x^3}}{\frac{x^3+1}{x^3}}$$

$$= \frac{\lim_{x\to\infty}\sqrt{\frac{9x^6-6}{x^6}}}{\lim_{x\to\infty}\left(1+\frac{1}{x^3}\right)} \qquad \left[\text{since } x^3 = \sqrt{x^6} \text{ for } x > 0\right]$$

$$= \frac{\lim_{x\to\infty}\sqrt{9-\frac{6}{x^6}}}{\lim_{x\to\infty}1+\lim_{x\to\infty}\left(\frac{1}{x^3}\right)} = \frac{\sqrt{\lim_{x\to\infty}9-6\lim_{x\to\infty}\left(\frac{1}{x^6}\right)}}{1+0}$$

$$= \sqrt{9-0} = 3.$$

Problem 35. Find the limit or show that it does not exist: $\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$.

$$= \lim_{x\to\infty} \frac{\frac{1-e^x}{e^x}}{\frac{1+2e^x}{e^x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+2}$$

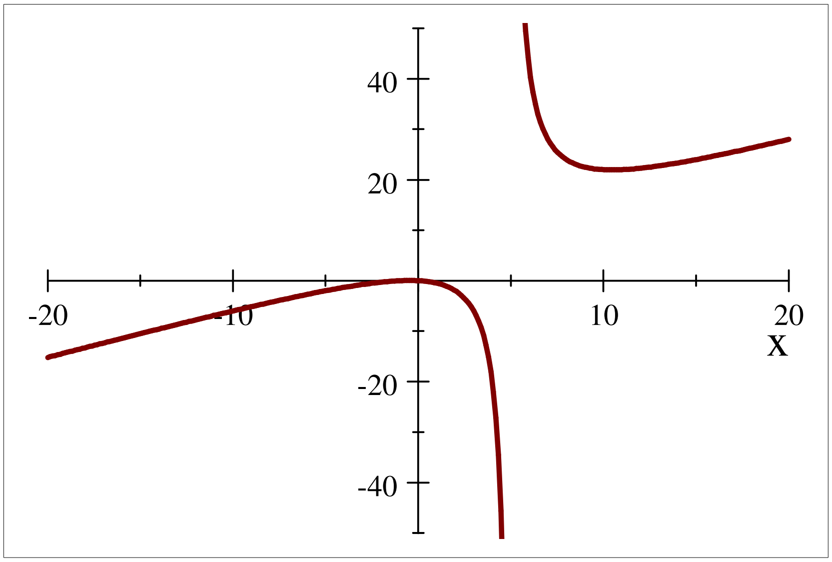
$$= \frac{0-1}{0+2} = -\frac{1}{2}.$$

Problem 45. Find the horizontal and vertical asymptotes of the curve $y = \frac{x^3-x}{x^2-6x+5}$. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

$$\frac{x^3-x}{x^2-6x+5} = \frac{x(x^2-1)}{(x-5)(x-1)}$$

$$= \frac{x(x-1)(x+1)}{(x-5)(x-1)} = \frac{x(x+1)}{x-5}$$

vertical at $x = 5$, and no horizontal since highest power of x is greater in numerator.



$$y = \frac{x^3 - x}{x^2 - 6x + 5}$$