MATH 1272: **Calculus II**

Discussion Instructor: Jodin Morey moreyjc@umn.edu
Discussion Session Website: math.umn.edu/~moreyjc

12.5 - Equations of Lines and Planes

Review:

Let's say we wish to define a line L in three dimensions. Furthermore, let's say we know a point on the line $P_0(x_0, y_0, z_0)$. We can therefore determine a vector $\vec{r}_0$ from the origin to P_0. If we also know of a vector $\vec{v} = \langle a, b, c\rangle$ which is parallel to L, we can then define a

Vector Equation of the Line L: $\langle x, y, z\rangle = \vec{r} = \vec{r}_0 + t\vec{v} = \left\langle x_0 + ta,\ y_0 + tb,\ z_0 + tc\right\rangle$. Each value of the parameter t gives a different position vector $\vec{r}$ for a point on L.

Parametric Equations of the Line L: $x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$

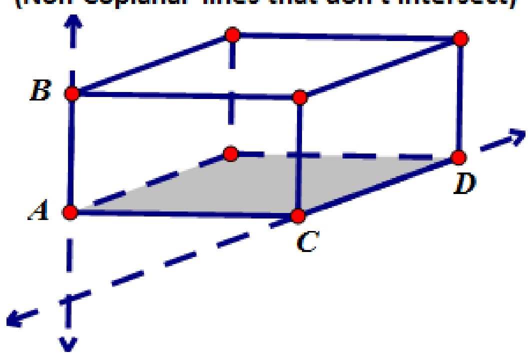
Direction Numbers of L: If $\vec{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, c are called direction numbers.

Symmetric Equations of L: If we solve for t in the parametric equations above, and then set these equal to each other, we have: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. This gives us another way to define the line, without using a parameter.

Parameterized Line Segment: The line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by the vector equation $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, where $0 \le t \le 1$.

Skew Lines

(Non-Coplanar lines that don't intersect)



It appears that these lines do intersect but it is a distortion in the 2-D drawing of the 3-D relationship - $\overline{AB}$ and $\overline{CD}$ will never intersect.

Skew Lines: Lines that do not intersect, and are not parallel (and therefore do not lie in the same plane).

Planes

A plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane.

Normal Vector: A vector $\vec{n}$ which is orthogonal to the plane.

Let $\vec{r}_0$ be the position vector of P_0, and let $\vec{n} = \langle a, b, c\rangle$ be a normal vector to the plane, then a **vector equation of the plane** is: $\langle a, b, c\rangle \cdot \left\langle x - x_0,\ y - y_0,\ z - z_0\right\rangle = \vec{n} \cdot \left(\vec{r} - \vec{r}_0\right) = 0$. Observe that this can also be written as the **scalar equation of the plane**: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. Finally, observe that we can rewrite this once again as a **linear equation**: $ax + by + cz + d = 0$, where $d = -(ax_0 + by_0 + cz_0)$. If at least one of a, b, c is nonzero, then this equation represents a plane with normal vector $\vec{n} = \langle a, b, c\rangle$.

Parallel Planes: To planes are parallel if their normal vectors are parallel. For example, $x + 2y - 3z + 4 = 0$ and $2x + 4y - 6z - 3$ are parallel because their normal vectors are $\vec{n}_1 = \langle 1, 2, -3\rangle$ and $\vec{n}_2 = \langle 2, 4, -6\rangle$, and we have $\vec{n}_2 = 2\vec{n}_1$.

If two planes are not parallel, they intersect in a straight line, and the angle between the two planes is defined as the acute angle θ between their normal vectors (recall that $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$).

Distance D from a point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$: $D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Problem #4 Find a vector equation and parametric equations for the line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t, \ \ y = 6 - 3t, \ \ z = 3 + 9t$.

The given parametric equation is associated with the following vector equation:
$\vec{v}(t) = \langle x_0 + 2t, \ y_0 - 3t, \ z_0 + 9t \rangle$, where $(x_0, y_0, z_0) = (-1, 6, 3)$.

So a vector equation parallel to this line, but with initial value $(0, 14, -10)$ would be $\vec{u}(t) = \langle 0 + 2t, \ 14 - 3t, \ -10 + 9t \rangle$.

And the parametric equations are: $x = 2t, \quad y = 14 - 3t, \quad z = -10 + 9t$.

Problem #12 Find parametric equations and symmetric equations for the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

$2y = -3z - x + 1, \qquad y = -\frac{3}{2}z - \frac{1}{2}x + \frac{1}{2}$
$z = y - x + 1$
$z = \left(-\frac{3}{2}z - \frac{1}{2}x + \frac{1}{2}\right) - x + 1$
$\frac{5}{2}z = -\frac{3}{2}x + \frac{3}{2}, \qquad z = -\frac{3}{5}x + \frac{3}{5}$.
$y = -\frac{3}{2}(-\frac{3}{5}x + \frac{3}{5}) - \frac{1}{2}x + \frac{1}{2} = \frac{2}{5}x - \frac{2}{5}$.
Observe that for $x = 0$, we have $y = -\frac{2}{5}$, and $z = \frac{3}{5}$.
Therefore, a parametric equation is:
$\langle t, \ -\frac{2}{5} + \frac{2}{5}t, \ \frac{3}{5} - \frac{3}{5}t \rangle$ where $(x_0, y_0, z_0) = (0, -\frac{2}{5}, \frac{3}{5})$.
Therefore, the symmetric equations are $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ or $\frac{x}{1} = \frac{y+\frac{2}{5}}{\frac{2}{5}} = \frac{z-\frac{3}{5}}{-\frac{3}{5}}$ or $x = \frac{5y+2}{2} = -\frac{5z-3}{3}$.

Problem #14 Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?

$\overrightarrow{AB} = \langle 1 + 2, \ 1 - 4, \ 1 - 0 \rangle = \langle 3, -3, 1 \rangle$.
$\overrightarrow{CD} = \langle 3 - 2, \ -1 - 3, \ -8 - 4 \rangle = \langle 1, -4, -12 \rangle$.

$\overrightarrow{AB} \cdot \overrightarrow{CD} = \langle 3, -3, 1 \rangle \cdot \langle 1, -4, -12 \rangle = 3 \cdot 1 + 3 \cdot 4 + 1(-12) = 3 \neq 0$.

So the lines are not perpendicular.

Problem #20 Given lines $L_1 : x = 5 - 12t, \ y = 3 + 9t, \ z = 1 - 3t$ and $L_2 : x = 3 + 8s, \ y = -6s, \ z = 7 + 2s$, determine whether the lines are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$\langle 5 - 12t, \ 3 + 9t, \ 1 - 3t \rangle, \qquad \langle 3 + 8s, \ -6s, \ 7 + 2s \rangle$
$|\langle -12, 9, -3 \rangle| = \sqrt{12^2 + 9^2 + 3^2} = 3\sqrt{26}$,
$|\langle 8, -6, 2 \rangle| = \sqrt{8^2 + 6^2 + 2^2} = 2\sqrt{26}$.
$\vec{u} = \frac{1}{\sqrt{26}}\langle -4, 3, -1 \rangle$ and $\vec{v} = \frac{1}{\sqrt{26}}\langle 4, -3, 1 \rangle$.
Observe that the unit vectors only differ by a sign, therefore they are parallel.

Problem #38 Find an equation of the plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$.

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
$5x + 4y - 2z = 0$ or $5(x - 0) + 4(y - 0) + (-2)(z - 0) = 0$, where $\vec{n} = \langle 5, 4, -2 \rangle$ and $(x_0, y_0, z_0) = (0, 0, 0)$.
$\overrightarrow{AB} = \langle -1 - 0, \ 3 - (-2), \ 1 - 5 \rangle = \langle -1, 5, -4 \rangle$.
Since $\vec{n}$ is normal to the given plane, which is perpendicular to our desired plane, $\vec{n}$ can be seen as a vector in our desired plane.

Therefore, the cross product of $\overrightarrow{AB} \times \vec{n}$ is orthogonal to the plane, and can be used as the normal vector for our desired plane (which is what we need to define the equation of the desired plane).

$$\overrightarrow{AB} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & -4 \\ 5 & 4 & -2 \end{vmatrix}$$

$$= (5(-2) - (-4)4)\vec{i} - ((-1)(-2) - (-4)5)\vec{j} + ((-1)4 - 5 \cdot 5)\vec{k}$$

$= 6\vec{i} - 22\vec{j} - 29\vec{k} = \langle 6, 22, 29 \rangle$.
Using $\vec{n}_1 = \langle 6, 22, 29 \rangle = \langle a, b, c \rangle$, $P_0 = (0, -2, 5) = (x_0, y_0, z_0)$, and the scalar equation of the plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, we have:

$$6x + 22(y + 2) + 29(z - 5) = 0.$$

Problem #52 Given the planes $2z = 4y - x$ and $3x - 12y + 6z = 1$, determine whether the planes parallel, perpendicular, or neither. If neither, find the angle between them.

We have $\vec{n}_1 = \langle 1, -4, 2 \rangle$ and $\vec{n}_2 = \langle 3, -12, 6 \rangle$.
$\langle 1, -4, 2 \rangle \cdot \langle 3, -12, 6 \rangle = 1 \cdot 3 + (-4)(-12) + 2 \cdot 6 = 63 \neq 0$.

So the planes are not perpendicular.

But observe that $3\vec{n}_1 = \vec{n}_2$, so they appear to be parallel.

Problem #62 Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

Consider the fact that the vector connecting these two points is orthogonal to the desired plane. Additionally, the point halfway along the segment connecting these two points is in the desired plane. And having obtained a normal vector, and a point in the plane, we can then use our equation for a plane.

The relevant vector is $\vec{n} = -\frac{1}{2}\left\langle -6 - 2,\ 3 - 5,\ 1 - 5 \right\rangle = \langle 4, 1, 2 \rangle$.

The point halfway along the segment connecting these points, can be found by $\frac{1}{2}\left(2 + (-6),\ 5 + 3,\ 5 + 1\right) = (-2, 4, 3)$.

Therefore, our equation is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or
$4(x - (-2)) + 1(y - 4) + 2(z - 3) = 4(x + 2) + (y - 4) + 2(z - 3) = 0$.

Problem #72 Find the distance from the point $(-6, 3, 5)$ to the plane $x - 2y - 4z = 8$.

Distance D from a point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$: $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

$x - 2y - 4z - 8 = 0, \qquad D = \frac{|1 \cdot (-6) + (-2)(3) + (-4)(5) + (-8)|}{\sqrt{1^2 + 2^2 + 4^2}} = \frac{|-6 - 6 - 20 - 8|}{\sqrt{21}} = \frac{40}{\sqrt{21}}$.

Problem #74 Find the distance between the parallel planes $6z = 4y - 2x$ and $9z = 1 - 3x + 6y$.

Observe that $(0, 0, 0)$ is a point in the first plane. Consider that every point in the first plane is the same distance from the second plane. Therefore, if we can find the distance between the origin and the second plane, we will have our answer.

$3x - 6y + 9z - 1 = 0$
Distance D from a point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$: $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3(0) + (-6)(0) + 9(0) + (-1)|}{\sqrt{3^2 + 6^2 + 9^2}} = \frac{1}{3\sqrt{14}}$.