**Discussion Instructor**: Jodin Morey moreyjc@umn.edu
**Discussion Session Website**: math.umn.edu/~moreyjc

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# 12.4 - Cross Product

## Review:

**Cross Product**: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector $\vec{a} \times \vec{b} = \left\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \right\rangle$.
Observe that, unlike the dot product, the result of the cross product is another vector, not a scalar. However, the cross product is only defined in three dimensions. One of the most important uses of the cross product, is that it produces a vector that is orthogonal to the given vectors.

**Determinant of Order 2**: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. For example, $\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14.$

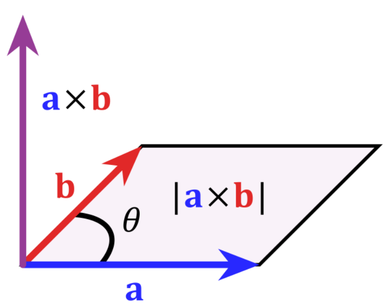
**Determinant of Order 3**: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$

Knowing this, we can now rewrite the definition of $\vec{a} \times \vec{b}$ as

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} - \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$= \left\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \right\rangle.$$

**Cross Product Magnitude Theorem**: If $\theta$ is the angle between $\vec{a}$ and $\vec{b}$ (with $0 \le \theta \le \pi$), then $\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta$.

Now we know the magnitude of $\vec{a} \times \vec{b}$, and that it is orthogonal to $\vec{a}$ and $\vec{b}$, but in which direction does it point? It turns out that the direction is given by the right-hand rule, curling your fingers from the vector $\vec{a}$ to $\vec{b}$.




**Length of the Cross Product**: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\left|\vec{a} \times \vec{b}\right|$ is equal to the area of the parallelogram $A$ determined by $\vec{a}$ and $\vec{b}$. (because $A = base \cdot height = |a| \cdot |b| \sin\theta = \left|\vec{a} \times \vec{b}\right|$)

**Parallel Vector Theorem**: To nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$

Observe that there are some algebraic properties of multiplication we are used to, that do not apply to the cross product. For example, the cross product is not generally commutative. In other words, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. Also, the cross product is not generally associative. In other words, $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$. So what properties do hold for cross product?

Properties of Cross Product: If $\vec{a}, \vec{b},$ and $\vec{c}$ are vectors and $d$ is a scalar, then

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ Anti-Commutative
- $(d\vec{a}) \times \vec{b} = d(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}),$ Scalar Multiplication
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c},$ Left Distribution over Addition
- $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c},$ Right Distribution over Addition
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c},$ Scalar Associativity
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.$ Vector Triple Product

All of the above properties can be proven to yourself by writing out the vectors in component form in carrying out the operations component wise on both sides of each equation.

# Triple Products

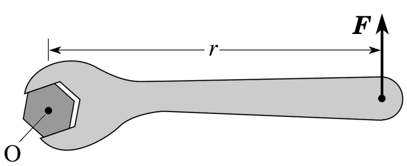
In addition to the Vector Triple Product defined above (which produces a vector), we have:

**Scalar Triple Product**: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, which produces a scalar.

Observe that the volume of a parallelepiped determined by the vectors $\vec{a}, \vec{b},$ and $\vec{c}$ is the magnitude of their scalar triple product: $\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$. If this volume is $0,$ it tells us that the vectors are contained in only one plane. We call this being **coplanar**.

# Torque

**Torque**: A force $\vec{F}$ acting on a point (of some rigid body) given by a position vector $\vec{r}$.
Relative to the origin, torque $\vec{\tau}$ is defined as the cross product $\vec{\tau} = \vec{r} \times \vec{F}$. The magnitude of the torque factor is $|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin\theta,$ where $\theta$ is the angle between the position vector $\vec{r}$ and the force vector $\vec{F}$.

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**Problem #2** Given $\vec{a} = \langle 1,1,-1 \rangle$ and $\vec{b} = \langle 2,4,6 \rangle,$ find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = (1 \cdot 6 - (-1)4)\vec{i} - (1 \cdot 6 - (-1)2)\vec{j} + (1 \cdot 4 - 1 \cdot 2)\vec{k}$$

$$= 10\vec{i} - 8\vec{j} + 2\vec{k} = \langle 10,-8,2 \rangle.$$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle 10,-8,2 \rangle \cdot \langle 1,1,-1 \rangle = 10 \cdot 1 - 8 \cdot 1 - 2 \cdot 1 = 0. \quad \checkmark$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle 10,-8,2 \rangle \cdot \langle 2,4,6 \rangle = 10 \cdot 2 - 8 \cdot 4 + 2 \cdot 6 = 0. \quad \checkmark$$

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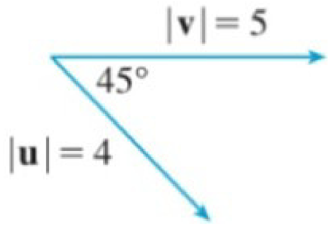


**Problem #14** Find $\left|\vec{u} \times \vec{v}\right|$ and determine whether $\vec{u} \times \vec{v}$ is directed into the page (screen) or out of the page (screen).

$\left|\vec{u} \times \vec{v}\right| = \left|\vec{u}\right|\left|\vec{v}\right| \sin\theta = 4 \cdot 5 \sin \frac{\pi}{2} = 20.$

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**Problem** #20 Find two unit vectors orthogonal to both $\vec{j} - \vec{k}$ and $\vec{i} + \vec{j}$.

$\vec{j} - \vec{k} = \langle 0,1,0\rangle - \langle 0,0,1\rangle = \langle 0,1,-1\rangle.$ $\quad \vec{i} + \vec{j} = \langle 1,1,0\rangle.$

$$\vec{j} - \vec{k} \times \vec{i} + \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= (1 \cdot 0 - (-1) \cdot 1)\vec{i} - (0 \cdot 0 - (-1)1)\vec{j} + (0 \cdot 1 - 1 \cdot 1)\vec{k}$

$= \vec{i} + \vec{j} - \vec{k} = \langle 1,1,-1\rangle.$

$\left|\vec{j} - \vec{k} \times \vec{i} + \vec{j}\right| = |\langle 1,1,-1\rangle| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}\,.$

Therefore, a unit vector orthogonal to the two given vectors is $\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$.

Pointing in the opposite direction, we also have the orthogonal unit vector $-\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$.