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11.10 - Taylor and MacLaurin Series

Review:

Taylor Series of f at a: If f has a power series representation (expansion) at a, that is, if f can be written as $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ when $|x-a| < R$, then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

In other words, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \ldots$.

Maclaurin Series: is simpler and very common. It is just a Taylor series of f at 0: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$
$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots$.

Note: Not all functions can be represented by a power series. If we are given a function f, how do we determine if it has a power series representation?

First let's define a few expressions: Let $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$. Observe that this is a polynomial which consists of the first $n+1$ terms of the Taylor series. It is referred to as the **nth-degree Taylor polynomial of f at a**. With this notation, we see that we can represent f as $f(x) = \lim_{n\to\infty} T_n(x)$. Let's also define the **remainder** of the Taylor series as $R_n(x) := f(x) - T_n(x)$. Therefore, $f(x) = T_n(x) + R_n(x)$.

Taylor Series Representation Theorem: Writing $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, then if $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$ then f is equal to the sum of its Taylor series on the interval $|x-a| < R$. [Theorem 8]

How do we show that $\lim_{n\to\infty} R_n(x) = 0$?

We will need:

Taylor's Inequality Theorem: If (for some n) $\left|f^{(n+1)}\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$ [Formula 9].
Also, observe that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x. [Equation 10]
You will see examples of how we use these facts to show $\lim_{n\to\infty} R_n(x) = 0$, and ultimately to then show that a function f is represented by its Taylor series in the problems below.

Useful Consequences of Above:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ for all a, x, and $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ for all x. [Equation 16]
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for $R = 1$.
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n}$ for $R = 1$.
$(1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\ldots(k-n+1)}{n!}x^n$
$= \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$

The equation above is called the binomial series, and $\binom{k}{n}$ ("k choose n") are the binomial coefficients. It is valid for any real numbers k and $|x| < 1$.

When power series are added or subtracted, we can do this the same way we do for polynomials. Multiplication and division also work the same as polynomial multiplication and division, except in each of these cases we restrict ourselves to the first few terms, enough to get the accuracy we wish to achieve.

Problem #4 Find the Taylor series for f centered at 4 if $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$. What is the radius of convergence of the Taylor series?

Since $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$, equation 6 in the text gives the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n(n+1)n!}(x-4)^n = \Sigma \frac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered at 4.

Apply the ratio test to find the radius of convergence R.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)} \bullet \frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty} \left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right|$$

$$= \frac{1}{3}|x-4| < 1$$

$$\Rightarrow \quad |x-4| < 3, \text{ so } R = 3.$$

Problem #8 Find the Maclaurin series for $f(x) = e^{-2x}$. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{-2x}	1
1	$-2e^{-2x}$	-2
2	$4e^{-2x}$	4
3	$-8e^{-2x}$	-8
4	$16e^{-2x}$	16
$\vdots$	$\vdots$	$\vdots$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \Sigma \frac{(-2)^n}{n!}x^n.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!} \bullet \frac{n!}{(-2)^n x^n}\right| = \lim_{n\to\infty} \frac{2|x|}{n+1} = 0$$

$$< 1 \text{ for all } x, \text{ so } R = \infty.$$

Problem #18 Find the Taylor series for $f(x) = \sin x$ centered at the value $a = \frac{\pi}{2}$. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{2})$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x-\frac{\pi}{2}\right)^k$$

$$= 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \frac{\left(x-\frac{\pi}{2}\right)^6}{6!} + \ldots$$

$= \sum_{n=0}^{\infty}(-1)^n \frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{\left|x-\frac{\pi}{2}\right|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{\left|x-\frac{\pi}{2}\right|^{2n}} \right] = \lim_{n\to\infty} \frac{\left|x-\frac{\pi}{2}\right|^2}{(2n+2)(2n+1)} = 0 < 1$ for all x, so $R = \infty$.

Problem #22 Prove that the series obtained in problem 18 (above) represents $\sin x$ for all x.

Set a real number x.
If $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$.

In each case, $|f^{(n+1)}(x)| \leq 1$ for all n, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \leq \frac{1}{(n+1)!}\left|x - \frac{\pi}{2}\right|^{n+1}$.

Recall: **Taylor's Inequality Theorem**: If (for some n) $\left|f^{(n+1)}\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$ [Formula 9].

Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10.
Recall: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x. [Equation 10]

So, $\lim_{n\to\infty} R_n(x) = 0$. And by Theorem 8, the series in Problem 18 represents $\sin x$ for all x.

Problem #34 Use a Maclaurin series from those listed in the review section to obtain the Maclaurin series for the function $f(x) = x^2 \ln(1+x^3)$.

Recall: $g(x) := \ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

So, $g(x^3) = \ln(1+x^3) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{3n}}{n}$,

and $f(x) = x^2 g(x^3) = x^2\ln(1+x^3) = \Sigma(-1)^{n-1}\frac{x^{3n+2}}{n}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{x^{3(n+1)+2}}{n+1}}{\frac{x^{3n+2}}{n}} \right| = \lim_{n\to\infty} \left|\frac{nx^3}{n+1}\right| = |x^3|$.

So we have $R = 1$.

Problem #44 Use the Maclaurin series for e^x to calculate $\frac{1}{\sqrt[10]{e}}$ correct to five decimal places.

$\frac{1}{\sqrt[10]{e}} = e^{-\frac{1}{10}}$ and recall that: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$,

So, $e^{-\frac{1}{10}} = 1 - \frac{1}{10} + \frac{\left(\frac{1}{10}\right)^2}{2!} - \frac{\left(\frac{1}{10}\right)^3}{3!} + \ldots$

Now, $1 - \frac{1}{10} + \frac{1}{2!\cdot 100} - \frac{1}{3!\cdot 1000} + \frac{1}{4!\cdot 10000} \approx 0.90484$, and subtracting $\frac{1}{5!\cdot 100000} \approx 8.3 \times 10^{-8}$ does not affect the fifth decimal place, so $e^{-\frac{1}{10}} \approx 0.90484$ by the alternating series estimation theorem (from 11.5).

Problem #64 Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}$.

$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$

$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, by Equation 16.