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# 11.9 - Representations of Functions as Power Series

## Review:

**Commonly Encountered Power Series**:
Observe that $(1-x)(1+x+x^2+x^3+\dots)$
$= (1+x+x^2+x^3+\dots) - (x+x^2+x^3+\dots) = 1.$
So, dividing both sides by $1-x$, we have: $\frac{1}{1-x} = 1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty} x^n.$    Convergent for $|x| < 1.$
(Notice how untrue the above calculations are for $x = 2$!!)

Calculation tricks: $\frac{1}{2+x} = \frac{1}{2}\frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}}$, and
$\frac{x^3}{1-x} = x^3\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}x^{n+3}.$

**Term-by-Term Differentiation and Integration Theorem**: If the power series $\Sigma c_n(x-a)^n$ has radius of convergence $R > 0$, then the function $f$ defined by $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable (and therefore continuous) on the radius $(a-R,\ a+R)$ and:

- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a) + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$,
- $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$

The radii of convergence of the power series in the above two equations are both $R$.
These two equations can be rewritten as:

- $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n]$,
- $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n dx.$

**Warning**: even though this theorem indicates that the *radius of convergence* remains the same, the endpoints may change as it relates to convergence. In other words, the *interval of convergence* may change upon taking a derivative or integrating.

**Problem #2**  Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2.$ What can you say about the series $\sum_{n=0}^{\infty}\frac{b_n}{n+1}x^{n+1}$ ? Why?

If $f(x) := \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2,2)$, then $\int f(x)dx = C + \sum_{n=0}^{\infty}\frac{b_n}{n+1}x^{n+1}$ has the same radius of convergence (by theorem 2), but may not have the same interval of convergence — it may happen that the integrated series converges at an endpoint, or both endpoints.

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**Problem #10**  Find a power series representation for the function $f(x) = \frac{x^2}{a^3-x^3}$ and determine the interval of convergence.

$$f(x) = \frac{x^2}{a^3}\cdot\frac{1}{1-\frac{x^3}{a^3}} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}.$$

The series converges when $\left|\frac{x^3}{a^3}\right| < 1 \quad\Rightarrow\quad |x^3| < |a^3|$

$\Rightarrow\quad |x| < |a|$, so $R = |a|$

and $I = (-|a|, |a|)$.

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**Problem #12**  Express the function $f(x) = \frac{x+2}{2x^2-x-1}$ as the sum of a power series by first using partial fractions. Find the interval of convergence.

$f(x) = \frac{x+2}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$

$x + 2 = A(x-1) + B(2x+1).$

Let $x = 1$ to get $3 = 3B \quad \Rightarrow \quad B = 1$

and $x = -\frac{1}{2} \quad \Rightarrow \quad \frac{3}{2} = -\frac{3}{2}A,$ or $A = -1.$

Thus, $\frac{x+2}{2x^2-x-1} = \frac{-1}{2x+1} + \frac{1}{x-1}$

$= -1\left(\frac{1}{1-(-2x)}\right) - 1\left(\frac{1}{1-x}\right)$

$= -\sum_{n=0}^{\infty}(-2x)^n - \sum_{n=0}^{\infty}x^n = -\sum_{n=0}^{\infty}[(-2)^n + 1]x^n.$

We represented $f$ as the sum of two geometric series; the first converges for $|2x| < 1$ or $x \in (-\frac{1}{2}, \frac{1}{2})$,

and the second converges for $(-1, 1)$.

Thus, the sum converges for $x \in (-\frac{1}{2}, \frac{1}{2}) = I.$

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**Problem** #20 Find a power series representation for the function $f(x) = \frac{x^2+x}{(1-x)^3}$ and determine the radius of convergence.

By example 5 in the text, we have: $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$, so
$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(n+1)x^n\right) \quad \Rightarrow \quad \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty}(n+1)nx^{n-1}.$

Thus, $f(x) = \frac{x^2+x}{(1-x)^3} = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^3} = \frac{x^2}{2} \cdot \frac{2}{(1-x)^3} + \frac{x}{2} \cdot \frac{2}{(1-x)^3}$

$= \frac{x^2}{2}\sum_{n=1}^{\infty}(n+1)nx^{n-1} + \frac{x}{2}\sum_{n=1}^{\infty}(n+1)nx^{n-1}$ (want to bring these under a common sum)

$= \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^{n+1} + \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n$ (want to bring these under a common sum)

$= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^n + \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n$ (make the exponents on $x$ equal by changing an index)

$= \sum_{n=2}^{\infty}\frac{n^2-n}{2}x^n + x + \sum_{n=2}^{\infty}\frac{n^2+n}{2}x^n$ (make the starting $n$ values equal)

$= x + \sum_{n=2}^{\infty}n^2x^n = \sum_{n=1}^{\infty}n^2x^n$, with radius ....

$R = 1.$

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**Problem #26** Evaluate the indefinite integral $\int \frac{t}{1+t^3}dt$ as a power series. What is the radius of convergence?

Observe that $\frac{t}{1+t^3} = t \cdot \left(\frac{1}{1-(-t^3)}\right) = t\sum_{n=0}^{\infty}(-t^3)^n = \sum_{n=0}^{\infty}(-1)^n t^{3n+1}$

Therefore, $\int \frac{t}{1+t^3}dt = \int \sum_{n=0}^{\infty}(-1)^n t^{3n+1}dt$

$= \sum_{n=0}^{\infty}(-1)^n \int t^{3n+1}dt = C + \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+2}}{3n+2}.$ Convergence?

The series for $\frac{1}{1+t^3}$ converges when $|-t^3| < 1 \quad \Rightarrow \quad |t| < 1,$ so $R = 1$ for that series and also for the series $\frac{t}{1+t^3}$.

By theorem 2, the series for $\int \frac{t}{1+t^3}dt$ also has $R = 1.$