MATH 1272: **Calculus II**

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11.8 - Power Series

Review:

A **power series** is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$ where x is a variable and the c_n's are constants called the coefficients of the series.

A **power series in** $(x-a)$, also called a **power series centered at** a, is a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$.

Recall: $(n+1)! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n \cdot (n+1) = n! \cdot (n+1)$.

Radius of Convergence Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

- The series converges only when $x = a$.
- The series converges for all x, or ...
- There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In the last case case above, R is called the **radius of convergence** of the power series. Note that this interval does not include the endpoints ($a+R$ or $a-R$). The power series may or may not converge at these points. They must be checked individually.

To determine the radius of convergence for a power series, the Ratio Test (or the Root Test) is often useful. To do this, we determine the values of x for which $\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| < 1$. It is this interval which is the radius of convergence. Examples for how to do this are in the problems below.

Problem #4 Find the radius-of-convergence and interval-of-convergence for the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$.

If $a_n := \frac{(-1)^n x^n}{\sqrt[3]{n}}$,

then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{(-1)^n x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1) x \sqrt[3]{n}}{\sqrt[3]{n+1}} \right| = |x| \lim_{n\to\infty} \left| \sqrt[3]{\frac{n}{n+1}} \right|$

$= |x| \lim_{n\to\infty} \sqrt[3]{\frac{1}{1+\frac{1}{n}}} = |x|$.

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ converges when $|x| < 1$, so $R = 1$.

When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the alternating series test.

When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges since it is a p-series ($p = \frac{1}{3} \leq 1$).

Thus, the interval of convergence is $(-1, 1]$.

Problem #28 Find the radius-of-convergence and interval-of-convergence for the series $\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}$.

If $a_n := \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2(n+1)-1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{n! x^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)x}{(2n+1)} \right| = |x| \lim_{n\to\infty} \left| \frac{n+1}{2n+1} \right|$$

$$= \frac{|x|}{2}.$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$ converges when $|x| < 2$, so $R = 2$.

When $x = \pm 2$, $|a_n| = \frac{n!2^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} 2^n$

$= \frac{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} 2^{n-1} > 2^{n-1} \geq 1$, so the series diverges by the divergence test.

Thus, the interval of convergence is $(-2, 2)$.

Problem #31 If k is a positive integer, find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$.

If $a_n := \frac{(n!)^k}{(kn)!} x^n$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{[(n+1)!]^k (kn)!}{(n!)^k [k(n+1)]!} |x|$

$$= \lim_{n\to\infty} \frac{(n+1)^k}{(kn+k)\cdot(kn+k-1)\cdot \ldots \cdot (kn+2)\cdot(kn+1)} |x|$$

$$= \lim_{n\to\infty} \left[\frac{n+1}{kn+1} \cdot \frac{n+1}{kn+2} \cdot \ldots \cdot \frac{n+1}{kn+k} \right] |x|$$

$$= \lim_{n\to\infty} \left[\frac{n+1}{kn+1} \right] \cdot \lim_{n\to\infty} \left[\frac{n+1}{kn+2} \right] \cdot \ldots \cdot \lim_{n\to\infty} \left[\frac{n+1}{kn+k} \right] |x|$$

$$= \left(\frac{1}{k} \right)^k |x|.$$

So, we have convergence when $\left(\frac{1}{k} \right)^k |x| < 1$ or $|x| < k^k$. And the radius of convergence is $R = k^k$.