MATH 1272: **Calculus II**

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11.6 - Absolute Convergence and the Ratio and Root Tests

Review:

Absolutely Convergent: A series Σa_n where the series of absolute values $\Sigma|a_n|$ is convergent.

Conditionally Convergent: A series Σa_n that is convergent but not absolutely convergent.

Ratio Test:

- If $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}| = L < 1,$ then the series Σa_n is absolutely convergent (and therefore convergent).
- If $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \infty,$ then the series Σa_n is divergent.
- If $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}| = 1,$ the ratio test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n.

Root Test:

- If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = L < 1,$ then the series Σa_n is absolutely convergent (and therefore convergent).
- If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \infty,$ then the series Σa_n is divergent.
- If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1,$ the root test is inconclusive.

Rearrangements: If Σa_n is an absolutely convergent series with sum $s,$ then any rearrangement of the terms of Σa_n produces the same sum. However, counterintuitively, if Σa_n is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of the terms of Σa_n that will produce a sum equal to r !

Problem #4 Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is absolutely convergent, conditionally convergent, or divergent.

Ler $b_n := \frac{n}{n^2+4} > 0,$ for $n \geq 1,$ and observe that $\{b_n\}$ is decreasing for $n \geq 2.$

Also: $\lim\limits_{n\to\infty} b_n = 0,$ so $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ converges by the alternating series test.

To determine absolute convergence, choose $a_n = \frac{1}{n}$ to get $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \lim\limits_{n\to\infty}\frac{\frac{1}{n}}{\frac{n}{n^2+4}} = \lim\limits_{n\to\infty}\frac{n^2+4}{n^2} = \lim\limits_{n\to\infty}\frac{1+\frac{4}{n^2}}{1} = 1 > 0.$

So, $\sum_{n=1}^{\infty}\frac{n}{n^2+4}$ diverges by the limit comparison test with the harmonic series.

Thus, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is conditionally convergent.

Problem #30 Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n}\frac{2^n n!}{5\cdot8\cdot11\cdot\ldots\cdot(3n+2)}$ is absolutely convergent, conditionally convergent, or divergent.

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(n+1)!}{5\cdot8\cdot11\cdot\ldots\cdot(3n+5)}}{\frac{2^n n!}{5\cdot8\cdot11\cdot\ldots\cdot(3n+2)}}\right| = \lim_{n\to\infty}\frac{2(n+1)}{3n+5}$$

$= \frac{2}{3} < 1,$ so the series converges absolutely by the ratio test.

Problem #32 A series Σa_n is defined by the equations $a_1 = 1$ and $a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$. Determine whether Σa_n converges or diverges.

By the recursive definition, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left| \frac{\frac{2+\cos n}{\sqrt{n}} a_n}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2+\cos n}{\sqrt{n}} \right|$$

$= 0 < 1,$ so the series converges absolutely by the ratio test.