MATH 1272: Calculus II

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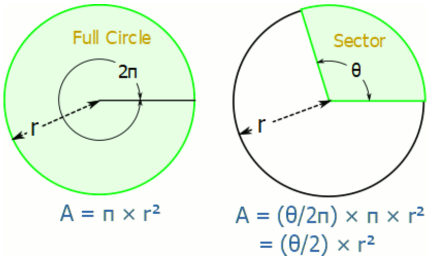
10.4 Areas and Lengths in Polar Coordinates

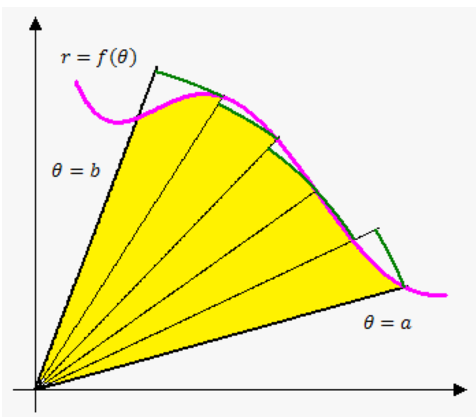
Review:

Recall that the area of a sector of a circle is $A = \frac{1}{2}r^2\theta$ where θ is the angle defining the sector.



We wish to determine the area swept out by the ray connecting the origin $(0,0)$ to our polar curve $r = f(\theta)$ as θ increases from $\theta_0 = a$ to $\theta_f = b$. We do it similarly to how we estimated the area under the curve using a Riemann approximation. We divide up the interval $[a,b]$ into n segments. Then, adjusting our area formula above, we have:

- Riemann Approximation: $A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$.
- Exact area: $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$.



Note that the above works for positive continuous $f(\theta)$, and intervals $[a,b]$ that are less than 2π. Determining the area for situations that fall outside these criteria is similarly achieved, but just requires a few slight alterations. Think about what these alterations might be!

Arc Length

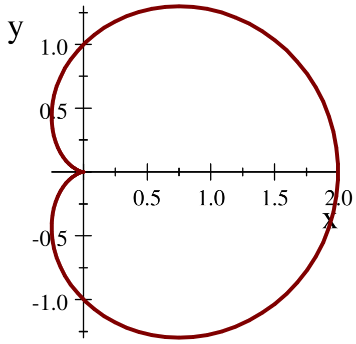
The polar arc length of $r = f(\theta)$ over $a \le \theta \le b$ is determined by altering the arc length rule from 10.2. Find the exact derivation in your text, but the result is: $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Problem #2 Find the area of the region that is bounded by the curve $r = \cos\theta$ and lies in the sector $0 \le \theta \le \frac{\pi}{6}$.

$$A = \int_0^{\frac{\pi}{6}} \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{6}} \cos^2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{1}{2}\sqrt{3}\right) = \frac{\pi}{24} + \frac{1}{16}\sqrt{3}.$$

Problem #6 Find the area of the region defined by the curve $r = 1 + \cos\theta$, with $0 \le \theta \le \pi$.



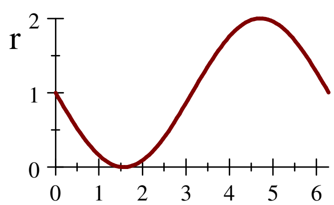
$r = 1 + \cos\theta$

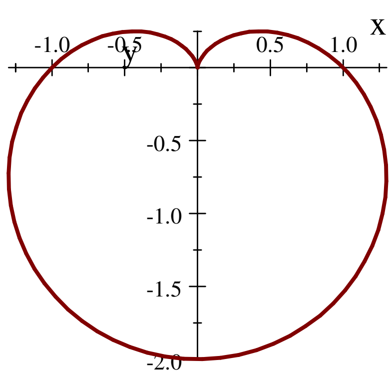
$$A = \int_0^{\pi} \tfrac{1}{2}(1 + \cos\theta)^2 d\theta = \tfrac{1}{2}\int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi}\left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right]d\theta = \tfrac{1}{2}\int_0^{\pi}\left(\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi} = \tfrac{1}{2}\left(\tfrac{3}{2}\pi + 0 + 0\right) - \tfrac{1}{2}(0) = \tfrac{3}{4}\pi.$$

Problem #10 Sketch the curve $r = 1 - \sin\theta$, and find the area that it encloses.



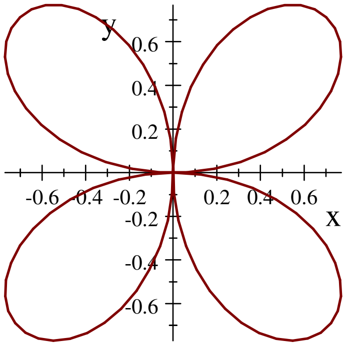


$$A = \int_0^{2\pi} \tfrac{1}{2}r^2 d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 - \sin\theta)^2 d\theta = \tfrac{1}{2}\int_0^{2\pi}(1 - 2\sin\theta + \sin^2\theta)d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}\left[1 - 2\sin\theta + \tfrac{1}{2}(1 - \cos 2\theta)\right]d\theta = \tfrac{1}{2}\int_0^{2\pi}\left(\tfrac{3}{2} - 2\sin\theta - \tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{1}{2}[(3\pi + 2) - 2] = \tfrac{3}{2}\pi.$$

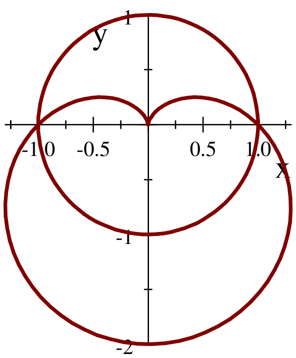
Problem #18 Find the area of the region enclosed by one loop of the curve $r^2 = \sin 2\theta$.



For $\theta = 0$ to $\theta = \frac{\pi}{2}$ the loop in the first quadrant is traced out by $r = \sqrt{\sin 2\theta}$.

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2\theta d\theta = \left[-\frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}.$$

Problem #24 Find the area of the region that lies inside the curve $r = 1 - \sin\theta$ and outside the curve $r = 1$.



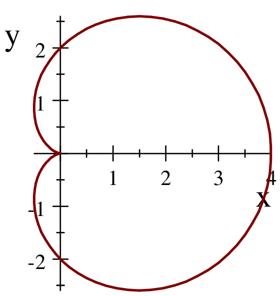
Where do we start and end our integration?
$1 - \sin\theta = 1 \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = 0$ or π.

$$A = \int_\pi^{2\pi} \frac{1}{2}\left[(1 - \sin\theta)^2 - 1^2\right] d\theta = \frac{1}{2}\int_\pi^{2\pi} (\sin^2\theta - 2\sin\theta) d\theta$$

$$= \frac{1}{4}\int_\pi^{2\pi} (1 - \cos 2\theta - 4\sin\theta) d\theta = \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta + 4\cos\theta\right]_\pi^{2\pi}$$

$$= \frac{1}{4}(2\pi - 0 + 4) - \frac{1}{4}(\pi - 0 - 4) = \frac{1}{4}\pi + 2.$$

Problem #48 Find the exact length of the polar curve $r = 2(1 + \cos\theta)$.



$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{[2(1 + \cos\theta)]^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

$$= \sqrt{8}\int_0^{2\pi}\sqrt{2\cdot\tfrac{1}{2}(1+\cos\theta)}\,d\theta = \sqrt{8}\int_0^{2\pi}\sqrt{2\cos^2\tfrac{\theta}{2}}\,d\theta$$

$$= \sqrt{8}\,\sqrt{2}\int_0^{2\pi}\left|\cos\tfrac{\theta}{2}\right|d\theta = 4\cdot 2\int_0^{\pi}\cos\tfrac{\theta}{2}\,d\theta \qquad \text{(by symmetry and } \sqrt{8}\,\sqrt{2} = 4\text{)}$$

$$= 8\left[2\sin\tfrac{\theta}{2}\right]_0^{\pi} = 8\left(2\sin\tfrac{\pi}{2} - 0\right) = 8(2) = 16.$$