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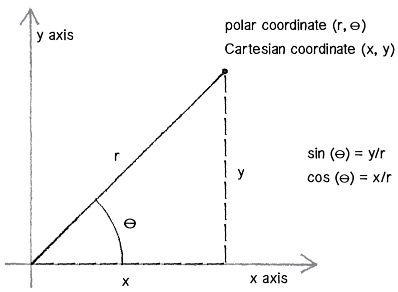
10.3 Polar Coordinates

Review:

Polar Coordinate System
When dealing with curvy lines, or areas/volumes bounded by curvy lines, it is often easier to do our desired calculations using a different coordinate system. The one we are used to is called the Cartesian coordinate system, but for curvy lines it's more convenient to use the polar coordinate system.

As you know, in the Cartesian coordinate system each point in the plane can be defined by is distance from the origin along the x-axis, and its distance from the origin along the y-axis, giving us the Cartesian coordinate (x, y).

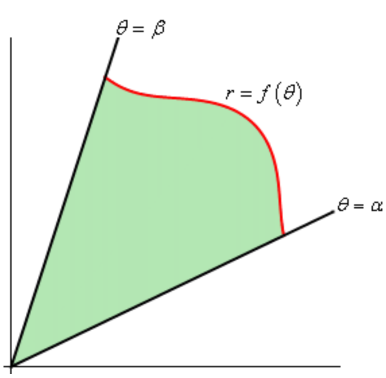


Similarly, in the polar coordinate system each point in the plane can be defined by its total distance (radius) from the origin r, and its angular distance θ (in radians) from the positive x-axis. We call the origin $(0,0)$ "**the pole**" when using the polar coordinate system, and we refer to the nonnegative part of the x-axis as the "**polar axis**." The resulting polar coordinates are therefore (r, θ). By convention, positive values of θ are associated with counterclockwise angular rotations from the polar axis, and negative values of θ with clockwise rotations. Obviously, if you add 2π to any angle, you end up back where you began. This means that (unlike the Cartesian coordinate system) a polar coordinate (r, θ) is not unique since you can add 2π to the angle, and it represents the same point $(r, \theta) = (r, \theta + 2\pi)$. Also, negative values of r are allowed, and have the following meaning: $(-r, \theta) = (r, \theta \pm k\pi)$ for any odd integer k.

Conversions:

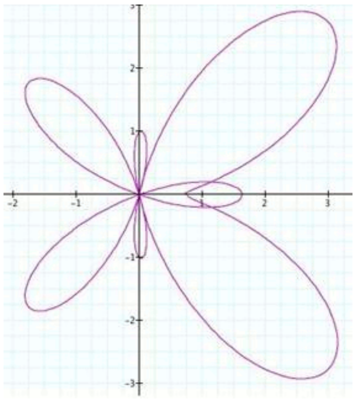
Obviously, it would be extremely useful to be able to convert from one coordinate system to the other. To this end, it is shown in the text (using some elementary trigonometry) that we have the following conversion from polar to Cartesian: $x = r\cos\theta$ and $y = r\sin\theta$.
And we also have this conversion from Cartesian to polar: $r^2 = x^2 + y^2$, and $\tan\theta = \frac{y}{x}$. Due to the ambiguity related to θ discussed above, you must be careful when determining θ with the above relation ($\theta = \tan^{-1}\frac{y}{x}$??). Particularly, ensure that your resulting θ leaves your coordinate (r, θ) in the same quadrant as your given point (x, y).

Defining curves in polar coordinates: $r = f(\theta)$ or $F(r, \theta) = 0$.

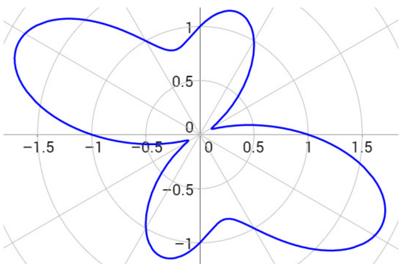


Symmetry

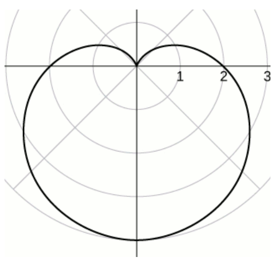
♦ If an equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the horizontal line defined by the polar axis (the x-axis).



♦ If the equation is unchanged when r is replaced by $-r$, or equivalently when θ is replaced by $\theta+\pi$, the curve is symmetric across the pole (the origin $(0,0)$). In other words, the graph would be unchanged if we were to rotate the graph by π (180°) around the origin.



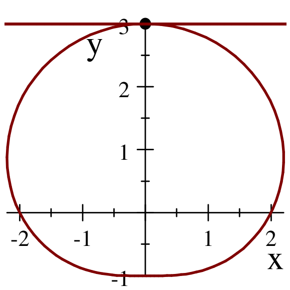
♦ If the equation is unchanged when θ is replaced by $\pi-\theta$, the curve is symmetric about the vertical line $\theta=\frac{\pi}{2}$ (the y-axis).

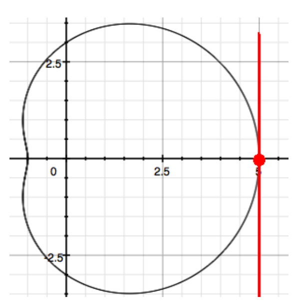


Tangents to Polar Curves

Let's say we want to find the Cartesian derivative $\frac{dy}{dx}$, but we are working with a polar curve $r(\theta)=f(\theta)$. Observe that we can rewrite our previous polar conversions as $x(\theta)=r(\theta)\cos\theta=f(\theta)\cos\theta$ and $y(\theta)=r(\theta)\sin\theta=f(\theta)\sin\theta$. Therefore, using the product rule and our previous slope equations from 10.2, we find that $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$.

Therefore, to find horizontal tangents, we want to find when $\frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta=0$ and $\frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta\neq 0$. Also, our vertical tangents will be when $\frac{dx}{d\theta}=0$ and $\frac{dy}{d\theta}\neq 0$.





Problem #6 Given Cartesian coordinates $\left(3\sqrt{3},3\right)$ and $(1,-2)$,

i) Find polar coordinates (r,θ) of each point, where $r > 0$ and $0 \le \theta < 2\pi$.
ii) Find polar coordinates (r,θ) of each point, where $r < 0$ and $0 \le \theta < 2\pi$.

Case: $(x,y) = \left(3\sqrt{3},3\right)$.

$$r = \sqrt{\left(3\sqrt{3}\right)^2 + 3^2} = \sqrt{27+9} = 6 \text{ and } \theta = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Since $\left(3\sqrt{3},3\right)$ is in the first quadrant, the polar coordinates are i) $(r,\theta) = (6, \frac{\pi}{6})$

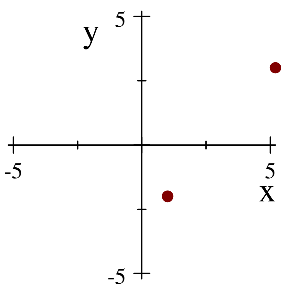
and ii) $\left(-r,\ \theta+\pi\right) = (6, \frac{\pi}{6} + \frac{6\pi}{6}) = \left(-6, \frac{7\pi}{6}\right)$.

Case: $(x,y) = (1,-2)$.
$r = \sqrt{1^2 + (-2)^2} = \sqrt{5}$ and $\theta = \tan^{-1}\left(\frac{-2}{1}\right) = \tan^{-1}(2)$.

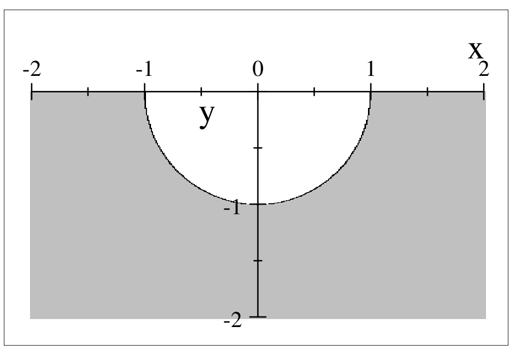
Since $(1,-2)$ is in the fourth quadrant, the polar coordinates are

i) $(r,\theta) = \left(\sqrt{5},\ 2\pi - \tan^{-1}(2)\right)$ and ...

ii) $(-r,\theta+\pi) = \left(-\sqrt{5},\ \pi - \tan^{-1}(2)\right)$.

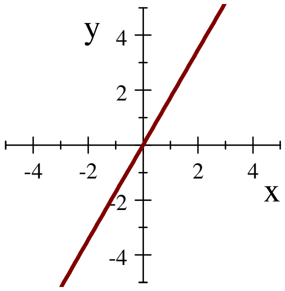


Problem #12 Sketch the region in the plan consisting of points whose polar coordinates satisfy the conditions: $r \ge 1,\quad \pi \le \theta \le 2\pi$.



Problem #18 Find a Cartesian equation for the curve: $\theta = \frac{\pi}{3}$.

$\tan\theta = \tan\frac{\pi}{3} \quad\Rightarrow\quad \frac{y}{x} = \sqrt{3} \quad\Rightarrow\quad y = \sqrt{3}\,x$, a line through the origin.

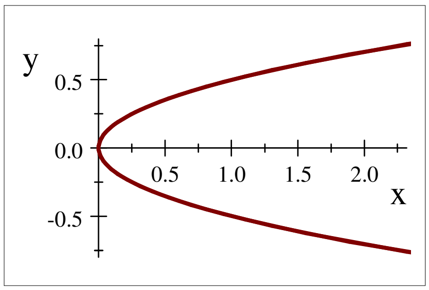


Problem #24 Find a polar equation for the curve represented by the Cartesian equation: $4y^2 = x$.

$4(r\sin\theta)^2 = r\cos\theta \quad \Rightarrow \quad 4r^2\sin^2\theta - r\cos\theta = 0$

$r(4r\sin^2\theta - \cos\theta) = 0 \quad \Rightarrow \quad r = 0 \text{ or } r = \frac{\cos\theta}{4\sin^2\theta} \quad \Rightarrow \quad r = 0 \text{ or } r = \frac{1}{4}\cot\theta\csc\theta.$

$r = 0$ is included in $\frac{1}{4}\cot\theta\csc\theta$ when $\theta = \frac{\pi}{2}$, so the curve is represented by the single equation $r = \frac{1}{4}\cot\theta\csc\theta$.



Problem #40 Sketch the curve with the polar equation $r = 2 + \sin\theta$ by first sketching the graph of r as a function of θ in Cartesian coordinates.