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10.2 Calculus with Parametric Curves

Review:

Let's say we wish to calculate the slope $\frac{dy}{dx}$, where $x = f(t)$ and $y = g(t)$ are differentiable functions of t. Additionally, assume y is a differentiable function of x. Since we can differentiate f and g with respect to t, we have $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Now, observe that with the chain rule we have $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. Therefore, assuming $\frac{dx}{dt} \neq 0$, we we can divide the previous equation by $\frac{dx}{dt}$ to get a **slope equation**: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Area Under the Curve

Recall that the area under the curve for $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geq 0$. If a curve is traced out by the parametric equations $x = f(t)$ and $y = g(t)$, on $\alpha \leq t \leq \beta$, the area under the curve (using the substitution rule for definite integrals) is (note: $dx = f'(t)dt$): $A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt.$

Arc Length

Expanding from our definition of arc length from section 8.1, using the substitution rule again with our parameterization, assuming $\frac{dx}{dt} \neq 0$, we have

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt} dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Surface Area of Rotated Curve

Similarly, adapting the formula from section 8.2 for surface area of our parametric curve rotated about the x-axis, we find the following formula: $S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Problem #2 Find $\frac{dy}{dx}$ for $x = \frac{1}{t}$ and $y = \sqrt{t}\, e^{-t}$.

$$\frac{dy}{dt} = t^{\frac{1}{2}}(-e^{-t}) + e^{-t}\left(\frac{1}{2}t^{-\frac{1}{2}}\right) = \frac{1}{2}t^{-\frac{1}{2}}e^{-t}(-2t+1) = \frac{-2t+1}{2t^{\frac{1}{2}}e^t}, \quad \frac{dx}{dt} = -\frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2t+1}{2t^{\frac{1}{2}}e^t}\left(-\frac{t^2}{1}\right) = \frac{(2t-1)t^{\frac{3}{2}}}{2e^t}.$$

Problem #4 Find an equation of the tangent to the curve $x = t - t^{-1}$, $y = 1 + t^2$, at the point corresponding to the value of the parameter $t = 1$.

$$\frac{dy}{dt} = 2t, \quad \frac{dx}{dt} = 1 + t^{-2} = \frac{t^2+1}{t^2}, \text{ and } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2t\left(\frac{t^2}{t^2+1}\right) = \frac{2t^3}{t^2+1}.$$

When $t = 1$, $(x,y) = (0,2)$

and $\frac{dy}{dx} = \frac{2}{2} = 1$.

So an equation of the tangent to the curve at the point corresponding to $t = 1$ is $y - 2 = 1(x - 0)$, or $y = x + 2$.

Problem #8 Find an equation of the tangent to the curve $x = 1 + \sqrt{t}$, $y = e^{t^2}$ at the point $(2, e)$ by two methods: **a**) **without**

eliminating the parameter.

$\frac{dy}{dt} = e^{t^2} \cdot 2t, \quad \frac{dx}{dt} = \frac{1}{2\sqrt{t}}$

and $\frac{dy}{dx} = \frac{2te^{t^2}}{\frac{1}{2\sqrt{t}}} = 4t^{\frac{3}{2}}e^{t^2}$.

So we have the slope of the tangent line for any t, now what do we need?

At $(2, e)$, we have $x = 1 + \sqrt{t} = 2 \quad \Rightarrow \quad \sqrt{t} = 1 \quad \Rightarrow \quad t = 1$, so $\frac{dy}{dx} = 4e$.

Finally, $y - e = 4e(x-2) \quad \Rightarrow \quad y = 4ex - 7e$.

b) **Find the equation by first eliminating the parameter**.

$\sqrt{t} = x - 1 \quad \Rightarrow \quad t = (x-1)^2$

So $y = e^{t^2} = e^{(x-1)^4}$.

To determine the line, we need a slope, so we calculate:
$y' = e^{(x-1)^4} \cdot 4(x-1)^3$.

At $(2, e)$, we have $y' = e \cdot 4 = 4e$.

So an equation of the tangent is $y - e = 4e(x-2)$, or $y = 4ex - 7e$.

Problem #14 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = t^2 + 1, \quad y = e^t - 1$. For which values of t is the curve concave upward?

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t}{2t}$

But what is $\frac{d^2y}{dx^2}$? It definitely is NOT $\left(\frac{dy}{dx}\right)^2$!! Nor is it $\frac{d}{dt}\left(\frac{dy}{dx}\right)$. You should understand it theoretically to be $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. But how do we take the derivative with respect to x, of a function in t? The key is recognizing that $\frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}$. Using this, we have:

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{2t} \frac{2te^t - e^t \cdot 2}{(2t)^2} = \frac{2e^t(t-1)}{(2t)^3} = \frac{e^t(t-1)}{4t^3}$. (Are we done?)

The curve is concave up when $\frac{d^2y}{dx^2} > 0$, that is, when $t > 0$ or $t > 1$.

Problem #18 Find the points on the curve $x = t^3 - 3t, \quad y = t^3 - 3t^2$ where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

$\frac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$, and $\frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$.

So $\frac{dy}{dt} = 0$ when $t = 0$ or 2.

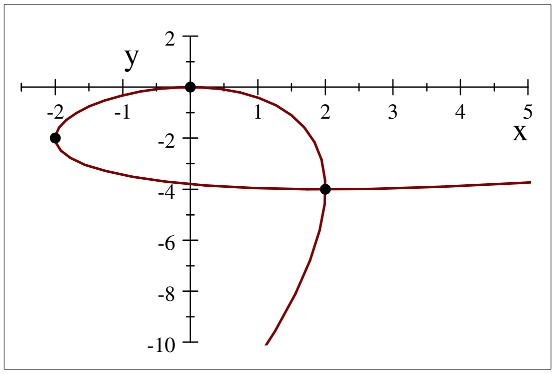
$\Rightarrow \quad (x, y) = (0, 0)$ or $(2, -4)$.

Also $\frac{dx}{dt} = 0$ when $t = -1$ or 1.

$\Rightarrow \quad (x, y) = (2, -4)$ or $(-2, -2)$.

The curve has horizontal tangents at $(0, 0)$ and $(2, -4)$, and vertical tangents at $(2, -4)$ and $(-2, -2)$.

The curve has both horizontal and vertical tangents at $(2,-4)$?!?



Problem #32 Find the area enclosed by the curve $x = t^2 - 2t, \quad y = \sqrt{t}$ and the y-axis.

The curve intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$.

The corresponding values of y are 0 and $\sqrt{2}$.

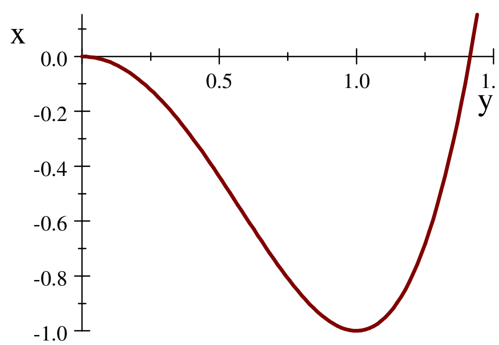
The area enclosed by the curve and the y-axis is given by: $\int_{y=0}^{y=\sqrt{2}} (x_R - x_L)dy$

$$= \int_{y=0}^{y=\sqrt{2}} (0 - x(y))dy$$

$$= \int_{t=0}^{t=2} [0 - x(t)]y'(t)dt$$

$$= -\int_0^2 (t^2 - 2t)\left(\tfrac{1}{2\sqrt{t}}dt\right) = -\int_0^2 \left(\tfrac{1}{2}t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)dt = -\left[\tfrac{1}{5}t^{\frac{5}{2}} - \tfrac{2}{3}t^{\frac{3}{2}}\right]_0^2$$

$$= -\left(\tfrac{1}{5} \cdot 2^{\frac{5}{2}} - \tfrac{2}{3} \cdot 2^{\frac{3}{2}}\right) = 2^{\frac{1}{2}}\left(\tfrac{4}{5} - \tfrac{4}{3}\right) = -\sqrt{2}\left(-\tfrac{8}{15}\right) = \tfrac{8}{15}\sqrt{2}.$$



Problem #42 Find the exact length of the curve $x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \le t \le 3$.

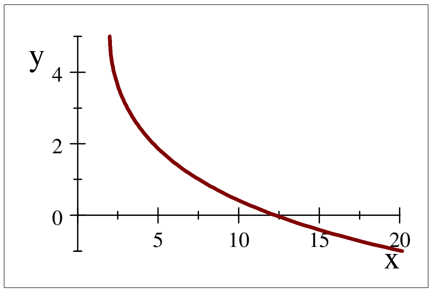
$\frac{dx}{dt} = e^t - e^{-t}$ and $\frac{dy}{dt} = -2$.

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^{2t} - 2 + e^{-2t}) + 4 = e^{2t} + 2 + e^{-2t}$.

Thus, $L = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt$

Observe that $e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$.

Thus, $L = \int_0^3 (e^t + e^{-t})dt = [e^t - e^{-t}]_0^3 = (e^3 - e^{-3}) - (1 - 1) = e^3 - e^{-3} \approx 20.$



Problem #52 Find the distance traveled by a particle with position $(x, y) = (\cos^2 t,\ \cos t)$ as t varies in the time interval $0 \le t \le 4\pi$. Compare with the length of the curve.

$$\text{(Distance Traveled)} = \int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2\cos t \sin t)^2 + (-\sin t)^2 = \sin^2 t(4\cos^2 t + 1).$$

$$\text{(Distance Traveled)} = \int_0^{4\pi} |\sin t| \sqrt{4\cos^2 t + 1}\, dt = 4\int_0^{\pi} \sin t \sqrt{4\cos^2 t + 1}\, dt$$

$$= -4\int_1^{-1} \sqrt{4u^2 + 1}\, du \qquad (\text{where } u = \cos t,\ du = -\sin t dt)$$

$$= 4\int_{-1}^{1} \sqrt{4u^2 + 1}\, du = 8\int_{u=0}^{u=1} \sqrt{4u^2 + 1}\, du$$

Let $2u = \tan\theta$ and $2du = \sec^2\theta d\theta$. Then:

$$\text{(Distance Traveled)} = 8\int_{\frac{\tan\theta}{2}=0}^{\frac{\tan\theta}{2}=1} \sqrt{\tan^2\theta + 1}\left(\frac{\sec^2\theta d\theta}{2}\right) = 4\int_{\theta=0}^{\theta=\tan^{-1}2} \sec^2\theta \sqrt{\sec^2\theta}\, d\theta$$

$$= 4\int_0^{\tan^{-1}2} \sec^3\theta d\theta .$$

The solution to this integral is given in the list in the back of the book by equation 71:

$$= 2[\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\tan^{-1}2}$$

$$= 2(\sec(\tan^{-1}2)\tan(\tan^{-1}2) + \ln|\sec(\tan^{-1}2) + \tan(\tan^{-1}2)|) - 2(\sec 0 \tan 0 + \ln|\sec 0 + \tan 0|)$$

$$= 2(2\sec(\tan^{-1}2) + \ln|\sec(\tan^{-1}2) + 2|) - 2(0 + \ln 1)$$

We wish to calculate $\sec t$, where $t = \tan^{-1}2$.

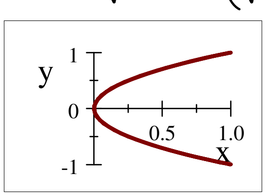
$2 = \frac{opposite}{adjacent}$. So we have that opposite $= 2$ and adjacent $= 1$.

Therefore, we have a right triangle in which hypotenuse $= \sqrt{2^2 + 1^2} = \sqrt{5}$.

So we can calculate $\sec(\tan^{-1}2) = \sec(t) = \frac{1}{\cos t} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{5}}{1} = \sqrt{5}$.

Therefore, we have: (Distance Traveled) $= 2\left(2\sqrt{5} + 2\ln\left(\sqrt{5} + 2\right)\right) - 0$

$$= 4\sqrt{5} + 2\ln\left(\sqrt{5} + 2\right) \approx 11.83.$$



Problem #62 Find the exact area of the surface obtained by rotating the curve $x = 3t - t^3$, $y = 3t^2$, on $0 \le t \le 1$ about the x-axis.

$$S = \int_{t=\alpha}^{t=\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2) + (6t)^2 = 9(1 + 2t^2 + t^4)$$

$$S = \int_0^1 18\pi t^2 \sqrt{1 + 2t^2 + t^4}\, dt$$

But observe that $1 + 2t^2 + t^4 = (1 + t^2)^2$. *So,*

$$S = \int_0^1 18\pi t^2 (1 + t^2) dt = 18\pi \int_0^1 (t^2 + t^4) dt = 18\pi \left[\frac{1}{3} t^3 + \frac{1}{5} t^5 \right]_0^1 = \frac{48}{5}\pi.$$