MATH 1272: **Calculus II**

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9.5 - Linear Equations

Review:

First-Order Linear Differential Equation: $\frac{dy}{dx} + P(x)y = Q(x)$
where P and Q are continuous on the interval in question.
Note: *it is not separable, so we cannot use our previous technique.*

Steps to solving:

- ♦ Determine the **Integrating Factor**: $I(x) = e^{\int P(x)dx}$.
- ♦ Multiply both sides of the differential equation by this integrating factor.
- ♦ Integrate both sides of the equation.

Application to Electric Circuits:
$L\frac{dI}{dt} + RI = E(t)$ where ...

- ♦ E is the voltage (force) generated by a battery or generator (in volts, V).
- ♦ I is the current through the circuit (in amperes, A),
- ♦ R is the resistance of the resistor (in ohms, Ω), and
- ♦ L is the inductance of an inductor (in units called henries, H).

Problem #12 Solve the differential equation: $x\frac{dy}{dx} - 4y = x^4e^x$.

$x\frac{dy}{dx} - 4y = x^4e^x \quad \Rightarrow \quad y' - \frac{4}{x}y = x^3e^x$.

$I(x) = e^{\int P(x)dx} = e^{\int\left(-\frac{4}{x}\right)dx} = e^{-4\ln|x|} = \left(e^{\ln|x|}\right)^{-4} = |x|^{-4} = x^{-4}$.

Multiplying the differential equation by $I(x)$, we have $x^{-4}y' - 4x^{-5}y = x^{-1}e^x$

$(x^{-4}y)' = \frac{e^x}{x} \quad \Rightarrow \quad x^{-4}y = \int\left(\frac{e^x}{x}\right)dx$

$\Rightarrow \quad y = x^4\left(\int\left(\frac{e^x}{x}\right)dx + C\right) = x^4(\mathrm{Ei}(x) + C)$, where $\mathrm{Ei}(x) := \int\left(\frac{e^x}{x}\right)dx$.

Problem #18 Solve the initial value problem: $2xy' + y = 6x, \quad x > 0, \quad y(4) = 20$.

$y' + \frac{1}{2x}y = 3$

$I(x) = e^{\int \frac{1}{2x}dx}$

$= e^{\frac{1}{2}\ln x}$

$= e^{\ln\left(x^{\frac{1}{2}}\right)} = \sqrt{x}$.

Multiplying by $\sqrt{x}$, we have $\sqrt{x}\,y' + \frac{1}{2\sqrt{x}}y = 3\sqrt{x}$

$\Rightarrow \quad (\sqrt{x}\,y)' = 3\sqrt{x} \quad \Rightarrow \quad \sqrt{x}\,y = \int 3\sqrt{x}\,dx = 2x^{\frac{3}{2}} + C$

$\Rightarrow \quad y = 2x + \frac{C}{\sqrt{x}}$. (Are we done?)

$y(4) = 20 \quad \Rightarrow \quad 8 + \frac{C}{2} = 20 \quad \Rightarrow \quad C = 24, \text{ so } y = 2x + \frac{24}{\sqrt{x}}.$

Problem #34 A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of $0.05\ g$ of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

How can we turn some of this into mathematically useful information?

Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds).

$y(0) = (0.05\ g/L)(400\ L) = 20\ g.$ (initial condition)

Recall the equation: $\frac{dy}{dt} = [Rate\ In] - [Rate\ Out]$, where [Rate of chlorine] = [rate fluid is flowing] • [concentration of chlorine in fluid].

And in order to determine the concentration for the outgoing flow, we will need to know the amount of total liquid in the tank at any particular moment.

The amount of liquid in the tank at times t is $(400 - 6t)\ L$ since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine in the tank at time t is $\frac{y(t)}{400-6t}\ \frac{g}{L}$. Also note that the tank is empty when $400 - 6t = 0$, or $t = \frac{200}{3} \approx 67$ sec.

Chlorine doesn't enter the tank (so $[Rate\ In] = 0$), but it leaves at a rate of $\left[10\ \frac{L}{s}\right]\left[\frac{y(t)}{400-6t}\ \frac{g}{L}\right] = \frac{10y(t)}{400-6t}\ \frac{g}{s}$.

Therefore, $\frac{dy}{dt} = -\frac{5y}{200-3t} \quad \Rightarrow \quad \int \frac{1}{ydy} = \int \frac{-5dt}{200-3t}.$

$$\ln y = \frac{5}{3}\ln(200 - 3t) + c$$

Now: $20 = y(0)$, so $\ln(20) = \frac{5}{3}\ln(200) + c \quad \Rightarrow \quad c = \ln(20) - \ln(200^{\frac{5}{3}}) = \ln\left(\frac{20}{200^{\frac{5}{3}}}\right).$

So, $y = e^{\ln\left((200-3t)^{\frac{5}{3}}\right) + \ln\left(\frac{20}{200^{\frac{5}{3}}}\right)}$

$= (200 - 3t)^{\frac{5}{3}} \cdot \frac{20}{200^{\frac{5}{3}}} = 20\left(\frac{200-3t}{200}\right)^{\frac{5}{3}} = 20\left(1 - \frac{3}{200}t\right)^{\frac{5}{3}}$ *gal* for $0 \le t \le 67\ s$, at which time the tank is empty.