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9.3 - Separable Equations

Review:

Separable equations are the easiest differential equations to solve. They are first order differential equations that you can manipulate into the following form: $\frac{dy}{dx} = g(x)f(y)$. That is, the product of two expressions, one in x, and one in y.

If $\frac{dy}{dx} = g(x)f(y)$ and $f(y) \neq 0$, then $\frac{1}{f(y)}dy = g(x)dx$.

Integrating, we get: $\int \frac{1}{f(y)}dy = \int g(x)dx$.

Define: $F(y(x)) := \int \frac{1}{f(y)}dy$ and $G(x) := \int g(x)dx$.

So, $F(y(x)) = G(x) + C$.

Looking at our conclusion above, you may ask yourself, "Have we solved the differential equation?" We certainly haven't solved it **explicitly** in terms of y.($y = f(x)$) When solving differential equations, the main goal is to eliminate all the derivatives, which we have done! It is also an important step (and we will be expecting it in this course) for you to solve any integrals generated in the process. Above, the differential equation has been solved, but only **implicitly** (because we don't have it in the form $y = f(x)$). If we can manipulate the equation further to isolate y on the left-hand side of the equation, we will have managed to solve it **explicitly**.

Say you have a family of curves as a solution to differential equation (which might have the form $y = cx^2$ where all the choices for c create the family). An **orthogonal trajectory** is a path through the slope field that intersects the family of curves at right angles. Below you see the family of curves $y = cx^2$ with the closed orthogonal trajectories drawn in.

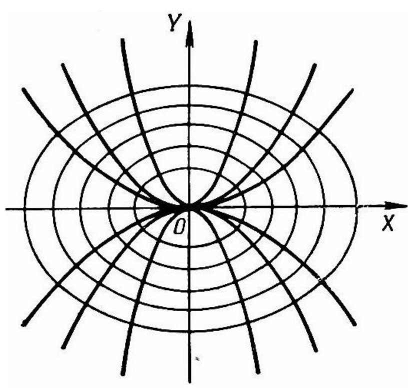
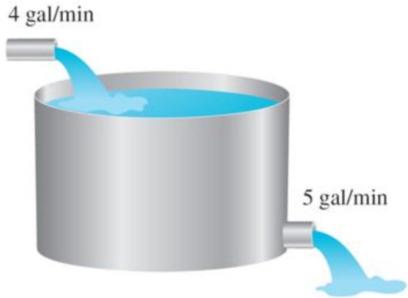


Fig. 106

In order for the path to be orthogonal, the path needs to intersect the family of curves at 90° angles. You may recall from earlier math classes that lines intersect in this way when their slopes are negative reciprocals of each other. So, since the differential equation can be interpreted as the slope of the function, once you have represented the original equation as $\frac{dy}{dx} = f(x)$, it is easy to solve for the orthogonal curves. Simply find the negative reciprocal, $-\frac{1}{f(x)}$ (assuming $f(x) \neq 0$), and solve the new differential equation to find the orthogonal family of curves.



Mixture Problems

You are given a problem where you have a tank of fluid, which has a certain amount of some substance (often times salt). Also, fluid is either entering or leaving the tank at a certain rate. It is your job to determine an equation for the amount of salt in the tank at any particular time. The process for solving for this equation proceeds as follows…

Set up the differential equation by writing...
$\frac{dx}{dt} = [Rate\ In] - [Rate\ Out]$ where these are the rates that the substance (not all the fluid) moves in or out of the tank.

So, our task is to determine $[Rate\ In]$ and $[Rate\ Out]$, after which we will be able to solve the differential equation as usual.

To determine the rate at which the substance is moving either in or out, we need to know how quickly the fluid is moving, and the amount of substance there is per unit of fluid (concentration).

For example:
$[Rate\ In] =$ [rate fluid is entering the tank] $\cdot$ [concentration of salt in this incoming fluid]. So we might have: $[Rate\ In] = [5\ gal/hr] \cdot [1.3\ lb/gal] = 6.5\ lb/hr$. Then,if you have fluid draining out you do the same thing for $[Rate\ Out]$. Now you can create your differential equation and solve it using a regular methods.

Problem #6 Solve the differential equation: $\frac{dv}{ds} = \frac{s+1}{sv+s}$.

Is it separable?

$(v+1)dv = (\frac{s+1}{s})ds$

$\int(v+1)dv = \int(1+\frac{1}{s})ds \Rightarrow \frac{1}{2}v^2 + v = s + \ln|s| + C$ [but we can do better]

$v^2 + 2v = 2s + 2\ln|s| + 2C$

$v^2 + 2v + 1 = 2s + 2\ln|s| + 2C + 1$

$(v+1)^2 = 2s + 2\ln|s| + 2C + 1$

$v + 1 = \pm\sqrt{2s + 2\ln|s| + K}$

$v = -1 \pm \sqrt{2s + 2\ln|s| + K}$, where $K = 2C + 1$.

Problem #16 Find the solution of the differential equation $\frac{dP}{dt} = \sqrt{Pt}$ that satisfies the initial condition $P(1) = 2$.

$\frac{dP}{\sqrt{P}} = \sqrt{t}\,dt$

$\int P^{-\frac{1}{2}}dP = \int t^{\frac{1}{2}}dt$

$2P^{\frac{1}{2}} = \frac{2}{3}t^{\frac{3}{2}} + C.$

We could solve for P from here, but you'll find that once you have solved your integrals, it's best to apply your initial conditions right away to simplify the rest of the process.

$P(1) = 2 \quad \Rightarrow \quad 2\sqrt{2} = \frac{2}{3} + C.$

So, $C = 2\sqrt{2} - \frac{2}{3}$, and $2P^{\frac{1}{2}} = \frac{2}{3}t^{\frac{3}{2}} + 2\sqrt{2} - \frac{2}{3}$.

Therefore, $\sqrt{P} = \frac{1}{3}t^{\frac{3}{2}} + \sqrt{2} - \frac{1}{3}$ and $P = \left(\frac{1}{3}t^{\frac{3}{2}} + \sqrt{2} - \frac{1}{3}\right)^2$.

Problem #20 Find the function f such that $f'(x) = f(x)(1 - f(x))$ and $f(0) = \frac{1}{2}$.

$\frac{dy}{dx} = y(1-y)$

$\frac{dy}{y(1-y)} = dx$, for $y \neq 0, 1$. $\quad \int \frac{dy}{y(1-y)} = \int dx$

$\Rightarrow \quad \int \left(\frac{A}{y} + \frac{B}{1-y}\right)dy = \int dx$

$A(1-y) + By = 1 \quad \Rightarrow \quad A + (B-A)y = 1 \quad \Rightarrow \quad A = 1$ and $B = 1$.

So, $\int \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int dx \quad \Rightarrow \quad \ln|y| - \ln|1-y| = x + c$.

Recall: $y(0) = \frac{1}{2}$

$\Rightarrow \quad \ln\left|\frac{1}{2}\right| - \ln\left|\frac{1}{2}\right| = 0 + c \quad \Rightarrow \quad c = 0.$ [Can we solve for y ?]

$\ln\left|\frac{y}{1-y}\right| = x \quad \Rightarrow \quad \left|\frac{y}{1-y}\right| = e^x \quad \Rightarrow \quad \frac{y}{1-y} = e^x.$

$y = (1-y)e^x = e^x - ye^x \quad \Rightarrow \quad y + ye^x = e^x$

$y(1+e^x) = e^x \quad \Rightarrow \quad y = \frac{e^x}{1+e^x}.$ (or equivalently $\frac{1}{1+e^{-x}}$).

Note that $y = 0$ and $y = 1$ not solution because they do not satisfy the initial condition $y(0) = \frac{1}{2}$.

Problem #30 For $y^2 = kx^3$, find the orthogonal trajectories of the family of curves. Use a graphing device to draw several members of each family on a common screen.

The curves $y^2 = kx^3$ form a family of power functions.

Differentiation gives $\frac{d}{dx}(y^2) = \frac{d}{dx}(kx^3) \quad \Rightarrow \quad 2yy' = 3kx^2$

$$\Rightarrow \quad y' = \frac{3kx^2}{2y} = \frac{3\left(\frac{y^2}{x^3}\right)x^2}{2y} = \frac{3y}{2x}.$$

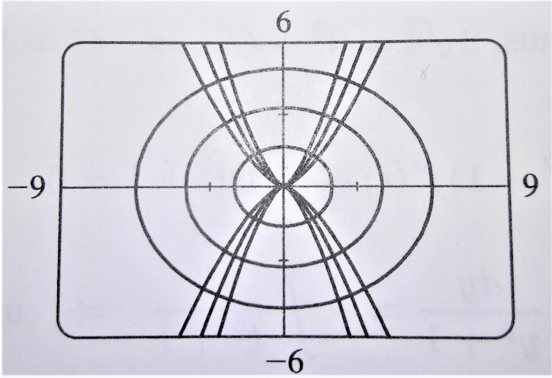
y' represents the slope of the tangent line at (x, y) on one of the curves in the family.

Thus, the orthogonal trajectories must satisfy: $y' = -\frac{2x}{3y}$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{3y} \quad \Rightarrow \quad 3y\,dy = -2x\,dx \quad \Rightarrow \quad \int 3y\,dy = \int -2x\,dx$$

$$\frac{3}{2}y^2 = -x^2 + c_1 \quad \Rightarrow \quad \frac{y^2}{6} = -\frac{x^2}{3} + c_2 \quad \Rightarrow \quad \frac{x^2}{3} + \frac{y^2}{6} = c_2.$$

This is a family of ellipses.



Random Problem: Find explicit particular solutions of the initial value problem:

$\frac{dy}{dx} = 2xy^2 + 3x^2y^2, \; y(1) = -1.$

$\frac{dy}{dx} = y^2(2x + 3x^2)$

$\frac{1}{y^2}\frac{dy}{dx} = 2x + 3x^2 \qquad \Rightarrow \qquad \int \frac{1}{y^2}\frac{dy}{dx}dx = \int(2x + 3x^2)dx$

$\Rightarrow -\frac{1}{y} = x^2 + x^3 + C, \qquad y(x) = -\frac{1}{x^2+x^3+C}.$ Are we done?

$-1 = -\frac{1}{1^2+1^3+C}, \qquad 2 + C = 1, \qquad C = -1.$

So, $y(x) = -\frac{1}{x^2+x^3-1}$.

Random Problem: Consider a reservoir with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily outflow of the (well-mixed) water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?

The volume of the lake is $8,000\ mft^3$.
Let $x(t)$ denote the amount of pollutants in the lake after t days,
measured in millions of cubic feet (mft^3).

The initial amount $x(0)$ of pollutants is...

$x_0 = (0.25\%)(8000) = 20\ mft^3.$
We want to know when $x(t) = (0.10\%)(8000) = 8\ mft^3$.
So now what?

We set up the differential equationa in infinitesimal form by writing...
$dx = [IN] - [OUT] = (0.0005)(500)dt - \frac{x}{8000} \cdot 500dt,$
which simplifies to $\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}$, or $\frac{dx}{dt} + \frac{1}{16}x = \frac{1}{4}$.

First, calculate the integrating factor: $\rho = e^{\int \frac{1}{16}dx} = e^{\frac{t}{16}}$.
Next, ...

Multiply both sides of the equation by the integrating factor
$e^{\frac{t}{16}}\left(\frac{dx}{dt} + \frac{1}{16}x\right) = \frac{1}{4}e^{\frac{t}{16}}$

Next, recognize the left-hand side of the resulting equation as the derivative of the product, $xe^{\frac{t}{16}}$.

Finally, integrate the equation: $D_t\left(xe^{\frac{t}{16}}\right) = \frac{1}{4}e^{\frac{t}{16}}$

$xe^{\frac{t}{16}} = \frac{1}{4}\int e^{\frac{t}{16}}dt = \frac{1}{4}\left(e^{\frac{t}{16}} \cdot 16\right) + C = 4e^{\frac{t}{16}} + C$

$x(t) = 4 + Ce^{-\frac{t}{16}}$

And then…

Plug-in our initial conditions: $x(0) = 4 + Ce^0 = 4 + C = 20,$

$\Rightarrow \qquad C = 16.$

$x(t) = 4 + 16e^{-\frac{t}{16}}$

"We want to know when $x(t) = 8\ mft^3$"

$8 = 4 + 16e^{-\frac{t}{16}}, \qquad \Rightarrow \qquad \frac{1}{4} = e^{-\frac{t}{16}}, \qquad \Rightarrow \qquad e^{\frac{t}{16}} = 4$

$\Rightarrow \qquad \ln 4 = \frac{t}{16}, \qquad \Rightarrow \qquad t = 16\ln 4 \approx 22.2$ days.