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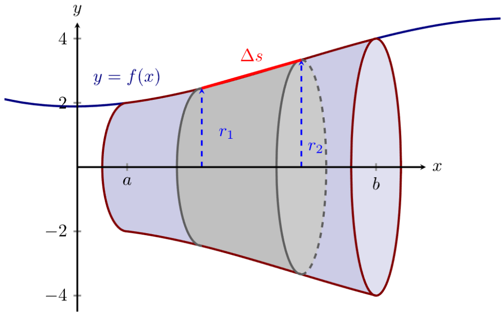
8.2 - Area of a Surface of Revolution

Review:



Review the beginning of the section which discusses how the following equations were derived. Doing this will assist you with the word problems in the homework, and on the exams.

The area of a surface of revolution is obtained by adding up the surface areas of slices of the figure, which the book refers to as "bands."

Each band (which is revolved about the axis) has a surface area of $2\pi rl$ where r is the average radius of the band ($r = \frac{1}{2}(r_1 + r_2)$ where r_1, r_2 are the radii at the endpoints of the band), and where l is the length of the band which we will determine using a line integral (l become ds in the integral). So if we call S the surface area, we have:

♦ $S = \int 2\pi r\,ds$ or ♦ $S = \int_a^b 2\pi f(x)\sqrt{1 + \left[f'(x)\right]^2}\,dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$

If we are integrating along the other axis, and therefore the function is of the form $x = g(y)$, then the equation becomes…

♦ $S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$

Problem #12 Find the exact area of the surface $x = 1 + 2y^2$, on $1 \le y \le 2$, obtained by rotating the curve about the x-axis.

$1 + \left(\frac{dx}{dy}\right)^2 = 1 + (4y)^2 = 1 + 16y^2.$

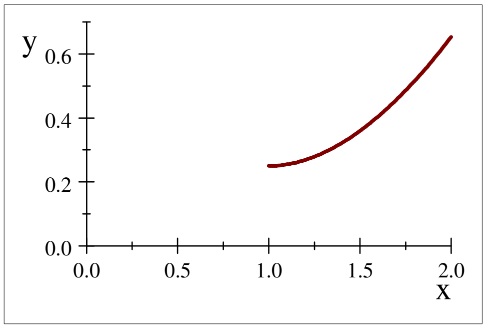
So, $S = 2\pi \int_1^2 y\sqrt{1 + 16y^2}\,dy$

Observe that if we wish to use $u = 1 + 16y^2$, then $du = 32y dy$, so we may wish to set it up as…

$S = \frac{\pi}{16}\int_1^2 \sqrt{1 + 16y^2}\,(32y dy) = \frac{\pi}{16}\int_1^2 \sqrt{u}\,du = \frac{\pi}{16}\left[\frac{2}{3}(16y^2 + 1)^{\frac{3}{2}}\right]_1^2$

$= \frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right) \approx 59.422.$

Problem #16 Let's say $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, on $1 \le x \le 2$ is rotated about the y-axis. Find the area of the resulting surface.



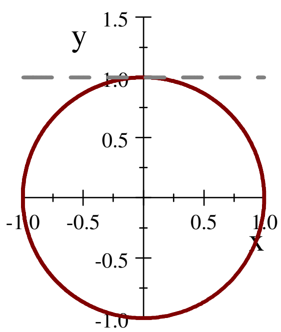
$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$ implies $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right) = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$

$= \left(\frac{x}{2} + \frac{1}{2x}\right)^2$!!!

So, $S = \int_1^2 2\pi x\sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\,dx = 2\pi\int_1^2 x\left(\frac{x}{2} + \frac{1}{2x}\right)dx = \pi\int_1^2 (x^2+1)dx = \pi\left[\frac{1}{3}x^3 + x\right]_1^2$

$= \pi\left[\left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)\right] = \frac{10}{3\pi}.$

Problem #33 Find the area of the surface obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = r$.



Example when $r = 1$

For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = \frac{-x}{\sqrt{r^2-x^2}}$.

The surface area generated is $S_1 = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2-x^2}}\,dx = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{\frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2}}\,dx$

$= 4\pi\int_0^r \left(r - \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2-x^2}}dx = 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2-x^2}} - r\right)dx.$

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \frac{x}{\sqrt{r^2-x^2}}$, so

$S_2 = \int_{-r}^{r} 2\pi\left(r + \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2-x^2}}\,dx = 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2-x^2}} + r\right)dx.$

Thus, the total area is $S = S_1 + S_2 = 8\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2-x^2}}\right)dx$

$= 8\pi r\int_0^r \left(\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}\right)dx = 8\pi r\left[r\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r$

$= 8\pi r\left(0 - r\frac{\pi}{2}\right) = 4\pi^2 r^2.$