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7.5 - Strategy for Integration

Review:

Putting it all together! In this section, you have to figure out which of the techniques of the previous sections to use for any particular integration problem. You also have to recall the various integration formulas. So, recall those from the review of section 7.1, in addition to the following…

$\int \sec x \; dx = \ln|\sec x + \tan x| + C$ $\int \csc x \; dx = \ln|\csc x - \cot x| + C$

$\int \frac{dx}{x^2-a^2} dx = \frac{1}{2a} \ln|\frac{x-a}{x+a}| + C$ $\int \frac{dx}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

General Strategies:

♦ Simplify the Integrand if possible
♦ Look for an obvious substitution
♦ Classify the integrand according to its form
  a) Trigonometric functions
  b) Rational functions
  c) Integration by parts
  d) Radicals
    i) $\sqrt{\pm x^2 \pm a^2}$, trigonometric substitution
    ii) $\sqrt[n]{ax+b}$, rationalizing substitution $u = \sqrt[n]{ax+b}$
♦ Try again
  a) Try seemingly unlikely substitutions
  b) Sometimes integration by parts can be used on single functions
  c) Manipulate the integrand, and perhaps something will present itself
  d) Think of similar previous problems which had a similar form
  e) Combine several of the methods above in succession.

Problem #4 Evaluate $\int \frac{\sin^3 x}{\cos x} dx$.

$$= \int \frac{\sin^2 x \sin x}{\cos x} = \int \frac{(1-\cos^2)\sin x}{\cos x} dx$$

$$= \int \frac{1-u^2}{u}(-du) \text{ (with } u = \cos x,\; du = -\sin x dx)$$

$$= \int (u - \tfrac{1}{u}) du = \tfrac{1}{2}u^2 - \ln|u| + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C.$$

Problem #48 Evaluate $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\, dx$

Let $u = \sqrt{1-x^2}$, so $u^2 = 1-x^2$, and $2udu = -2xdx$.

Then, $\int_{x=0}^{x=1} x\sqrt{2-\sqrt{1-x^2}}\, dx = \int_{u=1}^{u=0} \sqrt{2-u}\,(-udu)$.

Now let $v = \sqrt{2-u}$, so $v^2 = 2-u$, and $2vdv = -du$.

Thus, $\int_{u=1}^{u=0} \sqrt{2-u}\,(-udu) = \int_{v=1}^{v=\sqrt{2}} v(2-v^2)(2vdv) = \int_1^{\sqrt{2}} (4v^2 - 2v^4)dv = \left[\frac{4}{3}v^3 - \frac{2}{5}v^5\right]_1^{\sqrt{2}}$

$$= \left(\tfrac{8}{3}\sqrt{2} - \tfrac{8}{5}\sqrt{2}\right) - \left(\tfrac{4}{3} - \tfrac{2}{5}\right) = \tfrac{16}{15}\sqrt{2} - \tfrac{14}{15} \approx 0.575.$$

Problem #64 Evaluate $\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$.

Let $u = \sqrt{x} + 1$, so that $x = (u-1)^2$ and $dx = 2(u-1)du$.

Then $\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx = \int \frac{2(u-1)du}{\sqrt{u}} = \int \left(2u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du = \int \left(\frac{4}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$

$= \frac{4}{3}(\sqrt{x}+1)^{\frac{3}{2}} - 4\sqrt{\sqrt{x}+1} + C.$