Discussion Instructor: Jodin Morey moreyjc@umn.edu
Discussion Session Website: math.umn.edu/~moreyjc

7.3 - Trigonometric Substitutions

Review: Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Used to make inverse substitutions.

Problem #2 Evaluate $\int \frac{x^3}{\sqrt{x^2+4}}dx$ using the trigonometric substitution $x = 2\tan\theta$. Sketch and label the associated right triangle.

Let $x = 2\tan\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then, $dx = 2\sec^2\theta d\theta$.

So, $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\int \frac{x^3}{\sqrt{x^2+4}}dx = \int \frac{8\tan^3\theta}{2\sec\theta}(2\sec^2\theta d\theta) = 8\int \tan^3\theta\sec\theta d\theta$$

$$= 8\int \tan^2\theta\sec\theta\tan\theta d\theta = 8\int(\sec^2\theta - 1)\sec\theta\tan\theta d\theta$$

Let $u = \sec\theta$ and $du = \sec\theta\tan\theta$

So, $\int \frac{x^3}{\sqrt{x^2+4}}dx = 8\int(u^2-1)du = 8\left(\frac{1}{3}u^3 - u\right) + C = 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C.$

Observe the from above that $\sec\theta = \frac{\sqrt{x^2+4}}{2}$.

Therefore, $\int \frac{x^3}{\sqrt{x^2+4}}dx = 8\left(\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right) + C$

$$= 8\left(\frac{(x^2+4)^{\frac{3}{2}}}{3\cdot 8} - \frac{(x^2+4)^{\frac{1}{2}}}{2}\right) + C = \frac{1}{3}(x^2+4)^{\frac{3}{2}} - 4(x^2+4)^{\frac{1}{2}} + C.$$

Problem #30 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}dt$.

Let $u = \sin t,\ du = \cos t dt$. Then, $\int_{t=0}^{t=\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}dt = \int_{u=0}^{u=1} \frac{1}{\sqrt{1+u^2}}du$.

Now let $u = \tan\theta$. Then we have: $\sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$.

Also, $du = \sec^2\theta d\theta$. So then:

$$\int_{u=0}^{u=1} \frac{1}{\sqrt{1+u^2}}du = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{1}{\sec\theta}\sec^2\theta d\theta = \int_0^{\frac{\pi}{4}} \sec\theta d\theta$$

$= [\ \ln|\sec\theta + \tan\theta|\ ]_0^{\frac{\pi}{4}}$ (by (1) in section 7.2)

$= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right).$

Problem #41 A torus is generated by rotating the circle $x^2 + (y-R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

We use cylindrical shells and assume that $R > r$.

$x^2 = r^2 - (y-R)^2$ implies $x = \pm\sqrt{r^2 - (y-R)^2}$.

So $g(y) = 2\sqrt{r^2-(y-R)^2}$ and $V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2-(y-R)^2}\,dy$

$= \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du$ (where $u = y - R$)

$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du$ (where $u = r\sin\theta,\ du = r\cos\theta d\theta$)

$= 4\pi\left[-\frac{1}{3}(r^2-u^2)^{\frac{3}{2}}\right]_{-r}^{r} + 4\pi R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\cos^2\theta d\theta = -\frac{4\pi}{3}(0-0) + 4\pi R r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta d\theta$

$V = 2\pi R r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta = 2\pi R r^2\left[\theta + \frac{1}{2\sin}2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi^2 R r^2.$

Another Method: Use washers instead of shells, so $V = 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy,$ and evaluate integral using $y = r\sin\theta$.