Discussion Instructor: Jodin Morey moreyjc@umn.edu
Discussion Session Website: math.umn.edu/~moreyjc

7.2 - Trigonometric Integrals

Review: Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

a) If the power of cosine is odd $(n = 2k+1)$, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx.$$

Then, substitute $u = \sin x$.

b) If the power of sine is odd $(m = 2k+1)$, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx.$$

Then substitute $u = \cos x$. [Note that if the powers of **both** sine and cosine are odd, either a) or b) can be used.]

c) If the powers of both sine and cosine are even, use the half angle identities:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\qquad$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

It is also sometimes useful to use the identity: $\sin x \cos x = \frac{1}{2} \sin 2x$.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

a) If the power of secant is even $(n = 2k,\ k \geq 2)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx$$
$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx.$$

Then substitute $u = \tan x$.

b) If the power of tangent is odd $(m = 2k+1)$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx.$$

Then, substitute $u = \sec x$.

Evaluating: $\int \sin mx \cos nx \, dx$, $\quad \int \sin mx \sin nx \, dx$, $\quad \int \cos mx \cos nx \, dx$:

a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Other Useful Trigonometric Integrals:

$\int \tan x \, dx = \ln|\sec x| + C$ $\qquad$ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Problem #2 Evaluate $\int \sin^3\theta \cos^4\theta \, d\theta$.

$$\int \sin^3\theta \cos^4\theta \, d\theta = \int \sin^2\theta \cos^4\theta \sin\theta \, d\theta$$

$$= \int (1 - \cos^2\theta) \cos^4\theta \sin\theta \, d\theta$$

$u = \cos\theta, du = -\sin\theta d\theta$

$$= \int(1-u^2)u^4(-du) = \int(u^6-u^4)du = \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C = \tfrac{1}{7}\cos^7\theta - \tfrac{1}{5}\cos^5\theta + C.$$

Problem #46 Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1-\cos 4\theta}\, d\theta$.

$\int_0^{\frac{\pi}{4}} \sqrt{1-\cos 4\theta}\, d\theta$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1-(1-2\sin^2(2\theta))}\, d\theta = \int_0^{\frac{\pi}{4}} \sqrt{2\sin^2(2\theta)}\, d\theta = \sqrt{2}\int_0^{\frac{\pi}{4}} \sqrt{\sin^2(2\theta)}\, d\theta$$

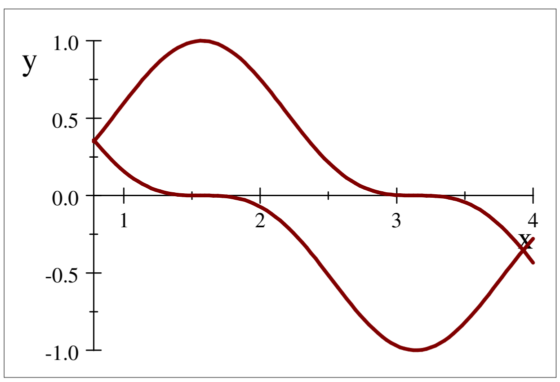
$$= \sqrt{2}\int_0^{\frac{\pi}{4}} |\sin(2\theta)| d\theta$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \sin 2\theta d\theta \qquad [\text{since } \sin 2\theta \geq 0 \text{ for } 0 \leq \theta \leq \tfrac{\pi}{4}]$$

$$= \sqrt{2}\left[-\tfrac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{4}} = \sqrt{2}\left(0-\tfrac{1}{2}\right) = \tfrac{\sqrt{2}}{2}.$$

Problem # 58 Find the area of the region bounded by the curves:

$$y = \sin^3 x, \quad y = \cos^3 x, \quad \tfrac{\pi}{4} \leq x \leq \tfrac{5\pi}{4}.$$



$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin^3 x - \cos^3 x)dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin^3 x dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos^3 x dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-\cos^2 x)\sin x dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1-\sin^2 x)\cos x dx$$

$$= \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} (1-u^2)(-du) - \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} (1-u^2)du$$

$$= 2\int_0^{\frac{\sqrt{2}}{2}} (1-u^2)du + 2\int_0^{\frac{\sqrt{2}}{2}} (1-u^2)du = 4\left[u - \tfrac{1}{3}u^3\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= 4\left[\left(\tfrac{\sqrt{2}}{2} - \tfrac{1}{3}\cdot\tfrac{\sqrt{2}}{4}\right) - 0\right] = 2\sqrt{2} - \tfrac{1}{3}\sqrt{2} = \tfrac{5}{3}\sqrt{2}.$$